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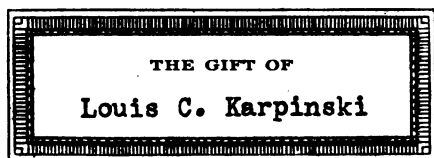
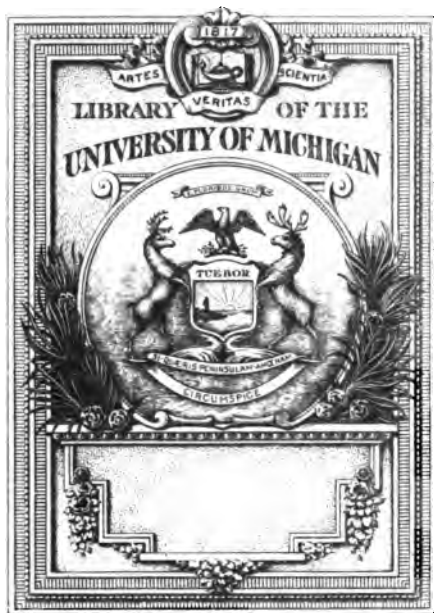
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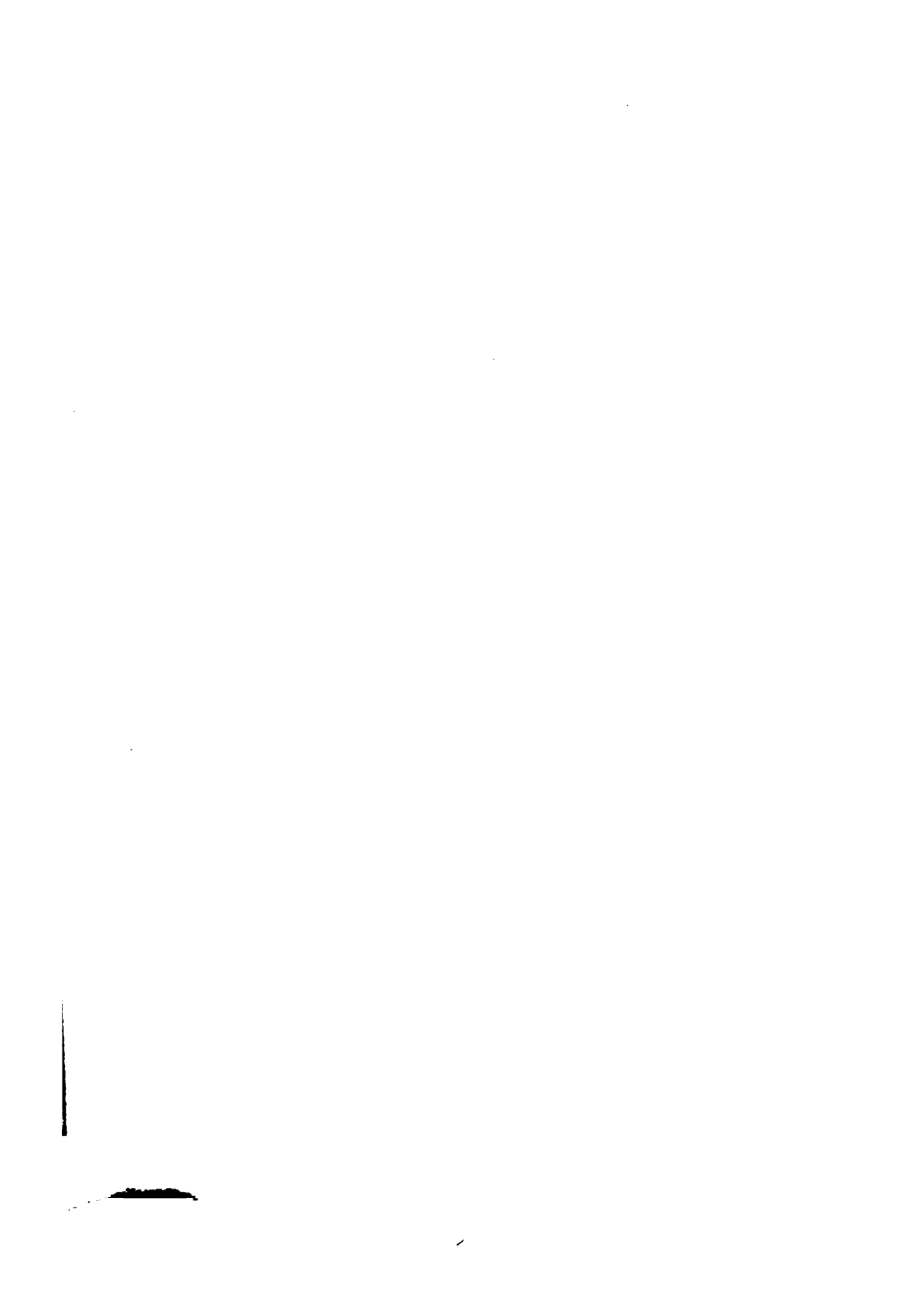


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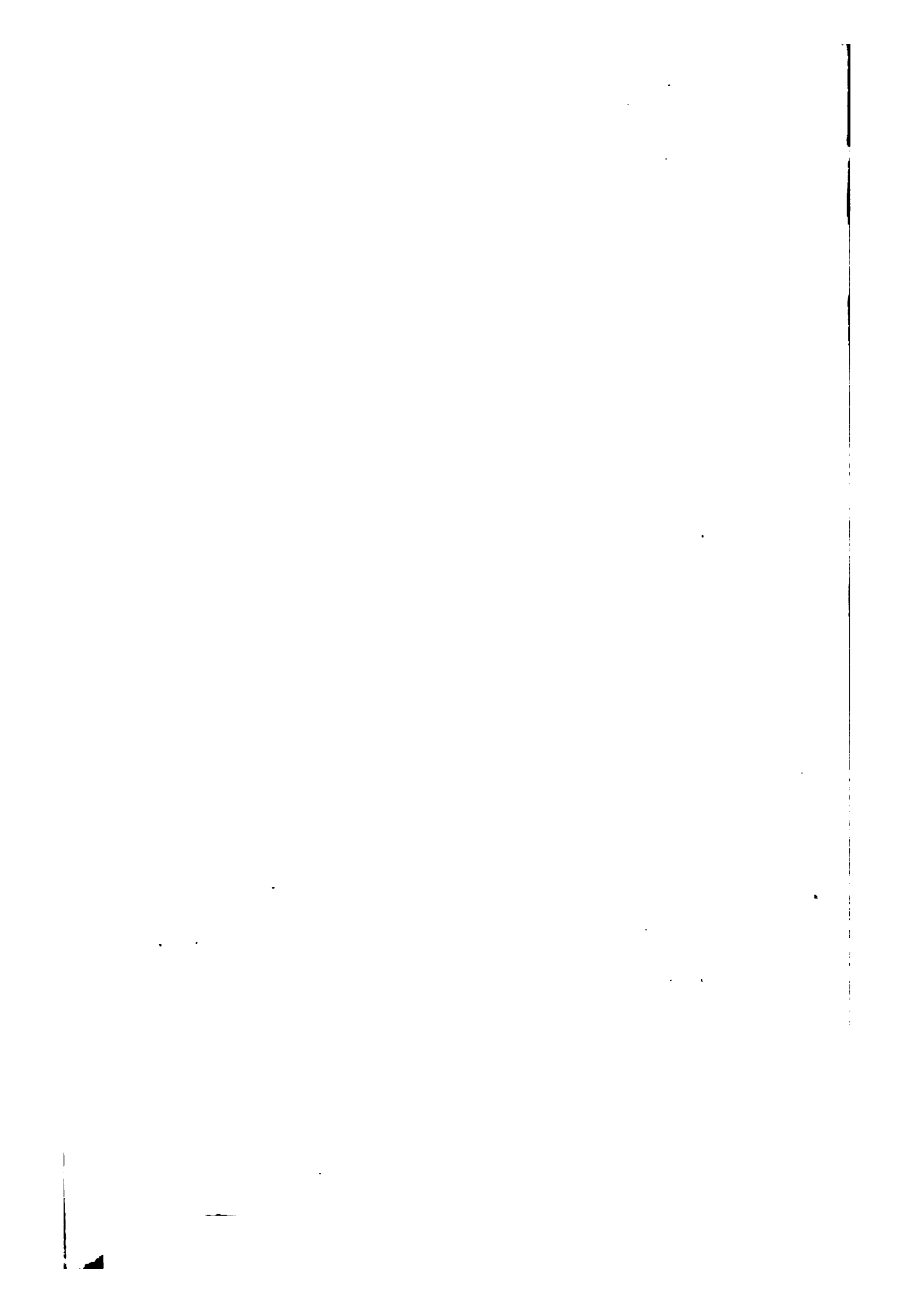
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## NOTE.

The Special Board for Mathematics in the University of Cambridge in a Report on Geometrical Teaching dated May 10, 1887, state as follows:

'The majority of the Board are of opinion that the rigid adherence to Euclid's texts is prejudicial to the interests of education, and that greater freedom in the method of teaching Geometry is desirable. As it appears that this greater freedom cannot be attained while a knowledge of Euclid's text is insisted upon in the examinations of the University, they consider that such alterations should be made in the regulations of the examinations as to admit other proofs besides those of Euclid, while following however his general sequence of propositions, so that no proof of any proposition occurring in Euclid should be accepted in which a subsequent proposition in Euclid's order is assumed.'

On March 8, 1888, Amended Regulations for the Previous Examination, which contained the following provision, were approved by the Senate :

'Euclid's definitions will be required, and no axioms or postulates except Euclid's may be assumed. The actual proofs of propositions as given in Euclid will not be required, but no proof of any proposition occurring in Euclid will be admitted in which use is made of any proposition which in Euclid's order occurs subsequently.'

And in the Regulations for the Local Examinations conducted by the University of Cambridge it is provided that :

'Proofs other than Euclid's will be admitted, but Euclid's Axioms will be required, and no proof of any proposition will be accepted which assumes anything not proved in preceding propositions in Euclid.'

**EUCLID'S**  
**ELEMENTS OF GEOMETRY**

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E U C L I D ' S  
ELEMENTS OF GEOMETRY

*EDITED FOR THE SYNDICS OF THE PRESS*

BY  
*Henry Martyn*  
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BOOKS I—VI, XI, XII.

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## PREFACE TO BOOKS I. AND II.

1-15-40.285  
IT was with extreme diffidence that I accepted an invitation from the Syndics of the Cambridge University Press to undertake for them a new edition of the Elements of Euclid. Though I was deeply sensible of the honour, which the invitation conferred, I could not but recognise the great responsibility, which the acceptance of it would entail.

The invitation of the Syndics was in itself, to my mind, a sign of a widely felt conviction that the editions in common use were capable of improvement. Now improvement necessitates change, and every change made in a work, which has been a text book for centuries, must run the gauntlet of severe criticism, for while some will view every alteration with aversion, others will consider that every change demands an apology for the absence of more and greater changes.

I will here give a short account of the chief points, in which this edition differs from the best known editions of the Elements of Euclid at present in use in England.

While the texts of the editions of Potts and Todhunter are confessedly little more than reprints of Simson's English version of the Elements published in 1756, the text of the present edition does not profess to be a *translation* from the Greek. I began by retranslating the First Book: but there proved to be so many points, in which I thought it

desirable to depart from the original, that it seemed best to give up all idea of simple translation and to retain merely the substance of the work, following closely Euclid's sequence of Propositions in Books I. and II. at all events.

Some of the definitions of Euclid, for instance *trapezium*, *rhomboid*, *gnomon* are omitted altogether as unnecessary. The word *trapezium* is defined in the Greek to mean "*any four sided figure other than those already defined*," but in many modern works it is defined to be "*a quadrilateral, which has one pair of parallel sides*." The first of these definitions is obsolete, the second is not universally accepted. On the other hand definitions are added of several words in general use, such as *perimeter*, *parallelogram*, *diagonal*, which do not occur in Euclid's list.

The chief alteration in the definitions is in that of the word *figure*, which is in the Greek text defined to be "*that which is enclosed by one or more boundaries*." I have preferred to define a *figure* as "*a combination of points, lines and surfaces*." That Euclid's definition leads to difficulty is seen from the fact that, though Euclid defines a circle as "*a figure contained by one line...*", he demands in his postulate that "*a circle may be described...*". Now it is the circumference of a circle which is described and not the surface. Again, when two circles intersect, it is the circumferences which intersect and not the surfaces.

I have rejected the ordinarily received definition of a square as "*a quadrilateral, whose sides are equal, and whose angles are right angles*." There is no doubt that, when we define any geometrical figure, we postulate the possibility of the figure; but it is useless to embrace in the definition more properties than are requisite to determine the figure.

The word axiom is used in many modern works as applicable both to simple *geometrical* propositions, such as "*two straight lines cannot enclose a space*," and to proposi-

tions, other than geometrical, accepted without demonstration and true *universally*, such as "*the whole of a thing is greater than a part.*" These two classes of propositions are often distinguished by the terms "geometrical axioms" and "general axioms." I prefer to use the word *axiom* as applicable to the latter class only, that is, to simple propositions, true of magnitudes of all kinds (for instance "*things which are equal to the same thing are equal to one another*"), and to use the term *postulate* for a simple *geometrical* proposition, whose truth we assume.

When a child is told that *A* weighs exactly as much as *B*, and *B* weighs exactly as much as *C*, he without hesitation arrives at the conclusion that *A* weighs exactly as much as *C*. His conviction of the validity of his conclusion would not be strengthened, and possibly his confidence in his conclusion might be impaired, by his being directed to appeal to the authority of the general proposition "*things which are equal to the same thing are equal to one another.*" I have therefore, as a rule, omitted in the text all reference to the general statements of axioms, and have only introduced such a statement occasionally, where its introduction seemed to me the shortest way of explaining the nature of the next step in the demonstration.

If it be objected that all axioms used should be clearly stated, and that their number should not be unnecessarily extended, my reply is that neither the Greek text nor any edition of it, with which I am acquainted, has attempted to make its list of axioms perfect in either of these respects. The lists err in excess, inasmuch as some of the axioms therein can be deduced from others: they err in defect, inasmuch as in the demonstrations of Propositions conclusions are often drawn, to support the validity of which no appeal can be made to any axiom in the lists.

Under the term *postulate* I have included not only what may be called the postulates of geometrical operation, such as "*it is assumed that a straight line may be drawn from any point to any other point,*" but also geometrical theorems, the truth of which we assume, such as "*two straight lines cannot have a common part.*"

➤ The postulates of this edition are nine in number.

Postulates 3, 4, 6 are the postulates of geometrical operation, which are common to all editions of the Elements of Euclid. Postulates 1, 5, 9 are the Axioms 10, 11, 12 of modern editions. Postulates 2, 7, 8 do not appear under the head either of axioms or of postulates in Euclid's text, but the substance of them is assumed in the demonstrations of his propositions.

Postulate 9 has been postponed until page 51, as it seemed undesirable to trouble the student with an attempt to unravel its meaning, until he was prepared to accept it as the converse of a theorem, with the proof of which he had already been made acquainted.

It may be mentioned that a proof of Postulate 5, "*all right angles are equal*" is given in the text (Proposition 10B), and that therefore the number of the Postulates might have been diminished by one: it was however thought necessary to retain this Postulate in the list, so that it might be used as a postulate by any person who might prefer to adhere closely to the original text of Euclid.

One important feature in the present edition is the greater freedom in the direct use of "the method of superposition" in the proofs of the Propositions. The method is used directly by Euclid in his proof of Proposition 4 of Book I., and indirectly in his proofs of Proposition 5 and of every other Proposition, in which the theorem of Proposition 4 is quoted. It seems therefore but a slight alteration to adopt the direct use of this method in the

proofs of any theorems, in the proofs of which, in Euclid's text, the theorem of Proposition 4 is quoted.

It may of course be fairly objected that it would be more logical for a writer, who uses with freedom the method of superposition, to omit the first three Propositions of Book I. To this objection my reply must be that it is considered undesirable to alter the numbering of the Propositions in Books I. and II. at all events. No doubt a work written merely for the teaching of geometry, without immediate reference to the requirements of candidates preparing for examination, might well omit the first three Propositions and assume as a postulate that "*a circle may be described with any point as centre, and with a length equal to any given straight line as radius,*" instead of the postulate of Euclid's text (Postulate 6 of the present edition), "*a circle may be described with any point as centre and with any straight line drawn from that point as radius.*"

The use of the words "each to each" has been abandoned. The statement that two things are equal to two other things each to each, seems to imply, according to the natural meaning of the words, that all four things are equal to each other. Where we wish to state briefly that  $A$  has a certain relation to  $a$ ,  $B$  has the same relation to  $b$ , and  $C$  has the same relation to  $c$ , we prefer to say that  $A, B, C$  have this relation to  $a, b, c$  **respectively**.

The enunciations of the Propositions in Books I. and II. have been, with some few slight exceptions, retained throughout, and the order of the Propositions remains unaltered, but different methods of proof have been adopted in many cases. The chief instances of alteration are to be found in Propositions 5 and 6 of Book I., and in Book II.

The use of what may be called impossible figures, such as occurred in Euclid's text in the proofs of Propositions

6 and 7 of Book I. has been avoided. It seems better to prove that a line cannot be drawn satisfying a certain condition without making a pretence of doing what is impossible.

Two Propositions (10 A and 10 B), have been introduced to shew that, if the method of superposition be used, we need not take as a postulate "*all right angles are equal to one another*," but that we may deduce this theorem from other postulates which have been already assumed.

Another new Proposition introduced into the text is Proposition 26 A, "*if two triangles have two sides equal to two sides, and the angles opposite to one pair of equal sides equal, the angles opposite to the other pair are either equal or supplementary*," which may be described, with reference to Euclid's text, as the missing case of the equality of two triangles. It is intimately connected with what is called in Trigonometry "the ambiguous case" in the solution of triangles.

Another new Proposition (41 A) is the solution of the problem "*to construct a triangle equal to a given rectilineal figure*." It appears to be a more practical method of solving the general problem of Proposition 45 "*to construct a parallelogram equal to a given rectilineal figure, having a side equal to a given straight line, and having an angle equal to a given angle*," to begin with the construction of a triangle equal to the given figure rather than to follow the exact sequence of Euclid's propositions.

In the notes a few "Additional Propositions" have been introduced containing important theorems, which did not occur in Euclid's text, but with which it is desirable that the student should become familiar as early as possible. Also outlines have been given of some of the many different proofs which have been discovered of Pythagoras's Theorem. They may be found interesting and useful as exercises for the student.

Euclid's proofs of many of the Propositions of Book II. are unnecessarily long. His use of the diagonal of the square in his constructions in Propositions 4 to 8 can scarcely be considered elegant.

It is curious to notice that Euclid after giving a demonstration of Proposition 1 makes no use whatever of the theorem. It seems more logical to deduce from Proposition 1 those of the subsequent Propositions which can be readily so deduced.

In Book II. outlines of alternative proofs of several of the Propositions have been given, which may be developed more fully and used in examinations, in place of the proofs given in the text. Some of these proofs are not, so far as I know, to be found in English text books. The most interesting ones are those of Propositions 12 and 13. Some, which I thought at first were new, I have since found in foreign text books.

The Propositions in the text have not been distinguished by the words "Theorem" and "Problem." The student may be informed once for all that the word **theorem** is used of a geometrical truth which is to be demonstrated, and that the word **problem** is used of a geometrical construction which is to be performed.

Although Euclid always sums up the result of a Proposition by the words  $\delta\pi\epsilon\rho\ \epsilon\delta\epsilon\iota\ \delta\epsilon\iota\chi\alpha\iota$  or  $\delta\pi\epsilon\rho\ \epsilon\delta\epsilon\iota\ \pi\omicron\iota\eta\sigma\alpha\iota$ , there seems to be no utility in putting the letters Q.E.D. or Q.E.F. at the end of a Proposition in an English text-book. The words "Quod erat demonstrandum" or "Quod erat faciendum" in a Latin text were not out of place.

When the book is opened, the reader will see as a rule on the left hand page a Proposition, and on the opposite page notes or exercises. The notes are either appropriate to the Proposition they face or introductory to the one next succeeding. The exercises on the right hand page are,

it is hoped, in all cases capable of being solved by means of the Proposition on the adjoining page and of preceding Propositions. They have been chosen with care and with the special view of inducing the student from the commencement of his reading to attempt for himself the solution of exercises.

For many Propositions it has been difficult to find suitable exercises: consequently many of the exercises have been specially manufactured for the Propositions to which they are attached. Great pains have been taken to verify the exercises, but notwithstanding it can scarcely be hoped that all trace of error has been eliminated.

It is with pleasure that I record here my deep sense of obligation to many friends, who have aided me by valuable hints and suggestions, and more especially to A. R. Forsyth, M.A., Fellow and Assistant Tutor of Trinity College, Charles Smith, M.A., Fellow and Tutor of Sidney Sussex College, R. T. Wright, M.A., formerly Fellow and Tutor of Christ's College, my brother-in-law the Reverend T. J. Sanderson, M.A., formerly Fellow of Clare College, and my brother W. W. Taylor, M.A., formerly Scholar of Queen's College, Oxford, and afterwards Scholar of Trinity College, Cambridge. The time and trouble ungrudgingly spent by these gentlemen on this edition have saved it from many blemishes, which would otherwise have disfigured its pages.

I shall be grateful for any corrections or criticisms, which may be forwarded to me in connection either with the exercises or with any other part of the work.

H. M. TAYLOR.

TRINITY COLLEGE, CAMBRIDGE,  
October 1, 1889.



## PREFACE TO BOOKS III AND IV.

**I**N Book III. the chief deviation from Euclid's text will be found in the first twelve Propositions, where a good deal of rearrangement has been thought desirable. This rearrangement has led to some changes in the sequence of Propositions as well as in the Propositions themselves; but, even with these changes, the first twelve Propositions will be found to include the substance of the whole of the first twelve of Euclid's text.

The Propositions from 13 to 37 are, except in unimportant details, unchanged in substance and in order.

The enunciation of the theorem of Proposition 36 has been altered to make it more closely resemble that of the complementary theorem of Proposition 35.

An additional Proposition has been introduced on page 186 involving the principle of the rotation of a plane figure about a point in its plane. It is a principle of which extensive use might with advantage be made in the proof of some of the simpler properties of the circle. It has not however been thought desirable to do more in this edition than to introduce the student to this method and by a selection of exercises, which can readily be solved by its means, to indicate the importance of the method.

The elegant theorem commonly called Ptolemy's Theorem is not found in Euclid's text. It was incorporated by Dr Robert Simson in one of his later editions as Proposition D of Book VI. The position in those modern editions which are based on Simson's conveys the idea that the methods of proportion are necessary for the proof of the theorem. I have been bold enough to insert it as Proposition 37 B in Book III, prefacing it by another theorem which appears as Proposition 37 A.

The additional Propositions at the end of Book III will introduce the reader to some of the simpler properties of Poles and Polars, of the Radical Axis of two Circles, of Orthogonal Circles, and of the Nine Point Circle of a Triangle.

Some of the definitions usually found in Book IV have been transferred in this edition to Book III. The change has been made chiefly with a view to convenience in the enunciation of exercises.

There is nothing which calls for special remark in Book IV.

I record with gratitude my obligations to those friends who have again so generously in this the second instalment of my work rendered me every assistance.

H. M. T.

TRINITY COLLEGE, CAMBRIDGE.

January 8, 1891.

## PREFACE TO BOOKS V. AND VI.

THE portion of Book V which is included in this edition contains only so much as is sufficient to establish in their generality those theorems in proportion which in Book VI are applied to geometrical magnitudes.

The method commonly used of representing the magnitudes under discussion by straight lines has been discarded as tending to mislead the student with respect to the general nature of the theorems of Book V, which are applicable to magnitudes of many kinds besides those met with in Geometry. In fact Book V is not essentially geometrical; a student whose knowledge of proportion is derived entirely from any of the standard treatises on Algebra is fully prepared to follow the applications of proportion to geometrical magnitudes as developed in Book VI.

In consequence of the omission of some of the Propositions of the Greek text, the Propositions in Book V have been renumbered; but for convenience the numbering

of the Greek text is printed in brackets at the head of each Proposition.

In Book VI a slight departure from Euclid's text is made in the treatment of similar figures. The definition of similar polygons which is adopted in this work brings into prominence the important property of the fixed ratio of their corresponding sides. Its use has the great merit of tending at once to simplicity and brevity in the proofs of many theorems.

The numbering of the Propositions in Book VI remains unchanged: Propositions 27, 28, 29 are omitted as in many of the recent English editions of Euclid, and in several cases a Proposition which consists of a theorem and its converse is divided into two Parts. Proposition 32 of Euclid's text, which is a very special case of no great interest, has been replaced by a simple but important theorem in the theory of similar and similarly situate figures.

The chief difficulty with respect to the additions which have been made to Book VI was the immense number of known theorems from which a selection had to be made.

I have attempted by means of two or three series of Propositions arranged in something like logical sequence to introduce the student to important general methods or well-known interesting results.

One series gives a sketch of the theory of transversals, and the properties of harmonic and anharmonic ranges and pencils, and leads up to Pascal's Theorem. Another series deals with similar and similarly situate figures and leads up

to Gergonne's elegant solution of the problem to describe a circle to touch three given circles. These are followed by an introduction to the method of Inversion, an account of Casey's extension of Ptolemy's Theorem, some of the important properties of coaxial circles, and Poncelet's Theorems relating to the porisms connected with a series of coaxial circles. /

No attempt has been made to represent the very large and still increasing collection of theorems connected with the "Modern Geometry of the Triangle."

I hereby acknowledge the great help I have received in this portion of my work from friends, and especially from Dr Forsyth and from my brother Mr J. H. Taylor. To the latter I am indebted for the Index to Books I—VI, which I hope may prove of some assistance to persons using this edition.

H. M. T.

TRINITY COLLEGE, CAMBRIDGE,  
March 16, 1893.



## PREFACE TO BOOKS XI. AND XII.

**I**N the text of Book XI. I have included only the first twenty-one Propositions of Euclid's Elements.

The substance as well as the order of the Propositions with the exception of Propositions 1, 2, 3, and 7 is the same as Euclid's Elements.

Proposition 2 of the Greek text implicitly assumes the truth of Proposition 3; Propositions 1, 3 and 7 of the Greek text become unnecessary when we adopt the definition of a plane which assumes that the straight line joining two points in the plane lies wholly in that plane.

The twenty-one Propositions are followed by nineteen Additional Propositions. Some of them cover the same ground as parts of Books XI. and XII. of the Greek text, which are now seldom read.

The Additional Propositions are followed by other matter in which the subjects discussed are treated in a less formal manner. Short expositions of the following subjects are included: Perspective drawing; the properties of tetrahedrons and parallelepipeds; the properties of the sphere and of systems of spheres; the problem to describe a sphere to touch four spheres; the elements of spherical trigonometry; the relations of spheres touching four faces of a tetrahedron; rotation in space, and the principles of geometrical reflection; the problem to describe a sphere to touch the four sides of a skew quadrilateral;

the construction of the five regular solid figures; and the determination of the surface and the volume of a sphere.

It is with the greatest satisfaction that I now complete the fourth and final part of the work which I undertook in the year 1887.

The chief portion of this instalment of my work was substantially completed before Easter 1894; but most of the remainder, which includes all the additional matter after page 575, has been thought out without any assistance from eyesight. I am indebted to friends for drawing the diagrams and preparing the text in this part of the book.

I have again to thank Dr Forsyth, my brothers, and my brother-in-law for their continued help. Some other friends have kindly given me assistance in this portion of my edition of Euclid; but I am especially indebted to my old College friend the Rev. A. M. Rendell, M.A., Vicar of St Margaret's, Leicester, who has generously sacrificed a month's holiday in order to help me to complete my work.

H. M. TAYLOR.

TRINITY COLLEGE, CAMBRIDGE.

*August 16, 1895.*



## HINTS TO THE READER.

GEOMETRY is a subject for the exercise of the understanding, not of the memory. The reader, who is beginning the study of geometry, is recommended not to attempt to learn by heart all the definitions and postulates before he proceeds to the propositions. As soon as he has made out the meaning of the enunciation and mastered the proof of a proposition he may then commit the enunciation to memory. He is also advised to refer constantly to the definitions for explanation of the meaning of any terms with which he is not familiar. The Index at the end of the book will be found useful for this purpose. The student is advised to make himself familiar with the practical use of geometrical constructions by means of a straight-edge and a pair of compasses.

## NOTE.

LITTLE is known about Euclid, the author of the Elements, beyond the fact that he taught at Alexandria during the reign of Ptolemy I. King of Egypt (323—285 B.C.).

Of the thirteen Books composing the Elements, Books VII.—X. and XIII. do not appear in this Edition. Books VII.—IX. treat of the properties of numbers, Book X. treats of incommensurable magnitudes and Book XIII. contains miscellaneous propositions in plane and solid geometry.

Euclid is supposed to have been also the author of the following works: the Data (a collection of geometrical theorems), the Phænomena (a treatise on geometrical astronomy), Optics, Reflections, and an Introduction to Harmony.

THE ELEMENTS OF GEOMETRY.

## BOOK I.

DEFINITION 1. *That which has position but not magnitude is called a point.*

The word *point* is used in many different senses. We speak in ordinary language of the point of a pin, of a pen or of a pencil. Any mark made with such a point on paper is of some definite size and is in some definite position. A small mark is often called a spot or a dot. Suppose such a spot to become smaller and smaller; the smaller it becomes the more nearly it resembles a *geometrical point*: but it is only when the spot has become so small that it is on the point of vanishing altogether, i.e. when in fact the spot still has position but has no magnitude, that it answers to the geometrical definition of a point.

A point is generally denoted by a single letter of the alphabet: for instance we speak of *the point A*.

DEFINITION 2. *That which has position and length but neither breadth nor thickness is called a line.*

*The extremities of a line are points.*

*The intersections of lines are points.*

The word *line* also is used in many different senses in ordinary language, and in most of these senses the main idea suggested is that of *length*. For instance we speak of a line of railway as connecting two distant towns, or of a sounding line as reaching from the bottom of the sea to the surface, and in so speaking we seldom think of the breadth of the railway or of the thickness of the sounding line.

When we speak of a *geometrical line*, we regard merely the length: we exclude the idea of breadth and thickness altogether: in fact we consider that the cross-section of the line is of no size, or in other words that the cross-section is a geometrical point.

If a point move with a continuous motion from one position to another, the path which it describes during the motion is a line.

DEFINITION 3. *That which has position, length and breadth but not thickness is called a surface.*

*The boundaries of a surface are lines.*

*The intersections of surfaces are lines.*

The word *surface* in ordinary language conveys the idea of extension in two directions: for instance we speak of the surface of the Earth, the surface of the sea, the surface of a sheet of paper. Although in some cases the idea of the thickness or the depth of the thing spoken of may be present in the speaker's mind, yet as a rule no stress is laid on depth or thickness. When we speak of a *geometrical surface* we put aside the idea of depth and thickness altogether. We are told that it takes more than 300,000 sheets of gold leaf to make an inch of thickness; but although the gold leaf is so thin, it must not be regarded as a geometrical surface. In fact each leaf however thin has always two bounding surfaces. The geometrical surface is to be regarded as absolutely devoid of thickness, and no number of surfaces put together would make any thickness whatever.

DEFINITION 4. *That which has position, length, breadth and thickness is called a solid.*

*The boundaries of solids are surfaces.*

DEFINITION 5. *Any combination of points, lines, and surfaces is called a figure.*

DEFINITION 6. *A line which lies evenly between points on it is called a straight line.*

This is Euclid's definition of a straight line. It cannot be turned to practical use by itself. We supplement the definition, as Euclid did, by making some assumptions the nature of which will be seen hereafter.

POSTULATES. There are a few geometrical propositions so obvious that we take the truth of them for granted, and a few geometrical operations so simple that we assume we may perform them when we please without giving any explanation of the process. The claim we make to use any one of these propositions, or to perform any one of these operations, is called a **postulate**.

POSTULATE 1. *Two straight lines cannot enclose a space.*

This postulate is equivalent to


*Two straight lines cannot intersect in more than one point.*

POSTULATE 2. *Two straight lines cannot have a common part.*

If two straight lines have two points  $A, B$  in common, they must coincide between  $A$  and  $B$ , since, if they did not, the two straight lines would enclose a space. Again, they must coincide beyond  $A$  and  $B$ , since, if they did not, the two straight lines would have a common part. Hence we conclude that

*Two straight lines, which have two points in common, are coincident throughout their length.*

Thus two points on a straight line completely fix the position of the line. Hence we generally denote a straight line by mentioning two points on it, and when the straight line is of finite length, we generally denote it by mentioning the points which are its two extremities.



For instance, if  $P$  and  $Q$  be two points on a straight line, the line is called the straight line  $PQ$  or the straight line  $QP$ , or sometimes more shortly  $PQ$  or  $QP$ : and the straight line which is *terminated* by two points  $P$  and  $Q$  is called in the same way  $PQ$  or  $QP$ .

It may be remarked that, when merely the actual length of the straight line is under discussion, we use  $PQ$  or  $QP$  indifferently: but that, when we wish to consider the direction of the line, we must carefully distinguish between  $PQ$  and  $QP$ .

**POSTULATE 3.** *A straight line may be drawn from any point to any other point.*

**POSTULATE 4.** *A finite straight line may be produced at either extremity to any length.*

The demands made in Postulates 3 and 4 are in practical geometry equivalent to saying that a 'straight edge' may be used for drawing a straight line from one point to another and for producing a straight line to any length.

We assume, as Euclid did, that it is possible to shift any geometrical figure from its initial position unchanged in shape and size into another position.

**Test of Equality of Geometrical Figures.** The criterion of the equality of two geometrical figures, which we shall use in most cases, is the possibility of shifting one of the figures, unchanged in shape and size, so that it exactly fits the place which the other of the figures occupies. (See Def. 21.)

This method of testing the equality of geometrical figures is generally known as **the method of superposition.**

**Test of equality of straight lines.** Two straight lines  $AB$ ,  $CD$  are said to be equal, when it is possible to shift either of them, say  $AB$ , so that it coincides with the other  $CD$ , the end  $A$  on  $C$  and the end  $B$  on  $D$ , or the end  $A$  on  $D$  and the end  $B$  on  $C$ .

**Addition of Lines.** Having defined the equality of straight lines, we proceed to explain what is meant by the addition of straight lines.



If in a straight line we take points  $A, B, C, D$  in order, we say that the straight line  $AC$  is the *sum* of the two straight lines  $AB, BC$  (or of any two straight lines equal to them),

and that the straight line  $AB$  is the *difference* of the two straight lines  $AC, BC$  (or of any two straight lines equal to them).

In the same way we say that the straight line  $AD$  is the sum of the three straight lines  $AB, BC, CD$ .

Again, if  $AB$  be equal to  $BC$ , we say that  $AC$  is double of  $AB$  or of  $BC$ .

**DEFINITION 7.** *A surface which lies evenly between straight lines on it is called a plane.*

This is Euclid's definition of a plane: there is the same difficulty in making use of it that there is in making use of his definition of a straight line.

Consequently this definition has by many modern editors been replaced by the following, which perhaps merely expresses Euclid's meaning in other words:

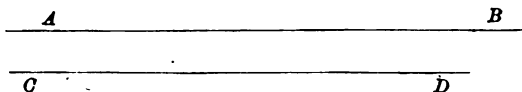
*A surface such that the straight line joining any two points in the surface lies wholly in the surface is called a plane.*

**DEFINITION 8.** *A figure, which lies wholly in one plane, is called a plane figure.*

All the geometrical propositions in the first six books of the Elements of Euclid relate to figures in one plane. This part of Geometry is called Plane Geometry.



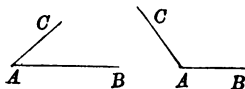
DEFINITION 9. *Two straight lines in the same plane, which do not meet however far they may be produced both ways, are said to be **parallel**\* to one another.*



DEFINITION 10. *A **plane angle** is the inclination to one another of two straight lines which meet but are not in the same straight line.*

The idea of an angle is one which it is very difficult to convey by the words of a definition. We will content ourselves by explaining some few things connected with angles.

If two straight lines  $AB$ ,  $AC$  meet at  $A$ , the amount of their divergence from one another or their inclination to one another is called *the angle which the lines make with one another or the angle between the lines, or the angle contained by the lines.*



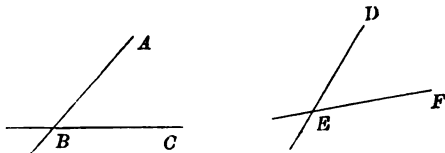
The angle formed by the straight lines  $AB$ ,  $AC$  is generally denominated  $BAC$ , or  $CAB$ , the middle letter always denoting the point where the lines meet, and the letters  $B$  and  $C$  denoting *any* two points in the straight lines  $AB$ ,  $AC$ . It must be carefully noted that the magnitude of the angle is not affected by the length of the straight lines  $AB$ ,  $AC$ .

The point  $A$ , where the two straight lines  $AB$  and  $AC$ , which form the angle  $BAC$ , meet, is called the **vertex** of the angle  $BAC$ .

If there be only two straight lines meeting at a point  $A$ , the angle formed by the lines is sometimes denoted by the single letter  $A$ .

\* Derived from  $\pi\alpha\rho\acute{\alpha}$  "by the side of" and  $\acute{\alpha}\lambda\lambda\eta\lambda\alpha\varsigma$  "one another":  $\pi\alpha\rho\acute{\alpha}\lambda\lambda\eta\lambda\omicron\iota\gamma\rho\alpha\mu\mu\alpha\iota$  "lines side by side",

**Test of Equality of Angles.** Two angles are said to be equal, when it is possible to shift the straight lines forming one of the angles, unchanged in position relative to each other, so as to exactly coincide in direction with the straight lines forming the other angle.



For instance, the angles  $ABC$ ,  $DEF$  will be equal, if it be possible to shift  $AB$ ,  $BC$  unchanged in position relative to each other, so that  $B$  coincides with  $E$ , and so that also either  $BA$  coincides in direction with  $ED$  and  $BC$  with  $EF$ , or  $BA$  coincides in direction with  $EF$  and  $BC$  with  $ED$ .

If a straight line move in a plane, while one point in the line remains fixed, the line is said to turn or revolve about the fixed point. If the revolving line move from any one position to any other position, it generates an angle, and the *amount of turning* from one position to the other is the measure of the magnitude of the angle between the two positions of the line.

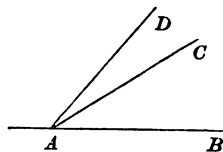
For instance each hand of a watch, as long as the watch is going, is turning uniformly round its fixed extremity, and is generating an angle uniformly.

This mode of regarding angles enables us to realize that angles are capable of growing to any size and need not be limited (as in most of the propositions in Euclid's Elements they are supposed to be) to magnitudes less than two right angles. (See Def. 11.)

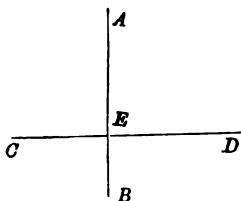
**Addition of Angles.** If three straight lines  $AB$ ,  $AC$ ,  $AD$  meet at the same point, we say that the angle  $BAD$  is the *sum* of the two angles  $BAC$ ,  $CAD$  (or of any two angles equal to them).

In the same way we say that the angle  $BAC$  is the *difference* of the two angles  $BAD$ ,  $CAD$  (or of any two angles equal to them).

Two angles such as  $BAC$ ,  $CAD$ , which have a common vertex and one common bounding line, are called **adjacent angles**.



**DEFINITION 11.** *If two adjacent angles made by two straight lines at the point where they meet be equal, each of these angles is called a right angle, and the straight lines are said to be at right angles to each other.*



*Either of two straight lines which are at right angles to each other is said to be perpendicular to the other.*

If a straight line  $AE$  be drawn from a point  $A$  at right angles to a given straight line  $CD$ , the part  $AE$  intercepted between the point and the straight line is commonly called the **perpendicular** from the point  $A$  on the straight line  $CD$ .

Euclid uses as a postulate,

**POSTULATE 5.** *All right angles are equal to one another.*

It is not necessary to assume this proposition, since it can be proved by the method of superposition. A proof will be found on a subsequent page. (p. 37)

**DEFINITION 12.** *An angle less than a right angle is called an acute angle.*

*An angle greater than a right angle and less than two right angles is called an obtuse angle.*

**DEFINITION 13.** *A line, which is such that it can be described by a moving point starting from any point of the line and returning to it again, is called a closed line.*

*A figure composed wholly of straight lines is called a rectilineal figure.*

*The straight lines, which form a closed rectilineal figure, are called the sides of the figure.*

*The sum of the lengths of the sides of any figure is called the perimeter of the figure.*

*The point, where two adjacent sides meet, is called a vertex or an angular point of the figure.*

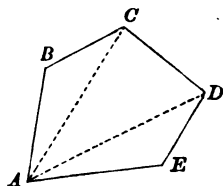
*The angle formed by two adjacent sides is called an angle of the figure.*

A straight line joining any two vertices of a closed rectilinear figure, which are not extremities of the same side, is called a **diagonal**\*.

The surface contained within a closed figure is called the **area** of the figure.

A closed rectilinear figure, which is such that the whole figure lies on one side of each of the sides of the figure, is called a **convex figure**.

A closed rectilinear figure is in general denoted by naming the letters, which denote its vertices, in order: for instance the five-sided figure in the diagram is denoted by the letters  $A, B, C, D, E$ , in order: i.e. it might be called the figure  $ABCDE$ , or the figure  $CBAED$ .



$A, B, C, D, E$  are its *vertices*.

$AB, BC, CD, DE, EA$  are its *sides*.

$ABC, BCD, CDE, DEA, EAB$  are its *angles*.

$AC, AD$  are two of its *diagonals*.

It will be observed that a closed figure has the same number of angles as it has sides.

If a closed figure have an even number of sides, we speak of a pair of sides as being opposite, and of a pair of angles as being opposite.

If a closed figure have an odd number of sides, we speak of an angle as being opposite to a side and *vice versa*.

For instance in the quadrilateral  $ABCD$  the side  $AD$  is said to be opposite to the side  $BC$ , and the angle  $BAD$  opposite to the angle  $BCD$ , but in the five-sided figure  $ABCDE$  the side  $CD$  is said to be opposite to the angle  $BAE$ , and the angle  $AED$  opposite to the side  $BC$ .

\* Derived from  $\delta\acute{\alpha}$  "through", and  $\gamma\omega\nu\alpha$  "an angle".

**DEFINITION 14.** *A figure, all the sides of which are equal, is called **equilateral**.*

*A figure, all the angles of which are equal, is called **equiangular**.*

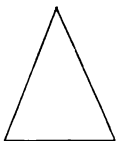
*A figure, which is both equilateral and equiangular, is called **regular**.*

**DEFINITION 15.** *A closed rectilineal figure, which has three sides\*, is called a **triangle**.*

*A closed rectilineal figure, which has four sides, is called a **quadrilateral**.*

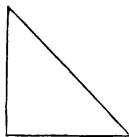
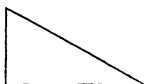
*A closed rectilineal figure, which has more than four sides, is called a **polygon**†.*

**DEFINITION 16.** *A triangle, which has two sides equal, is called **isosceles**‡.*



*A triangle, which has a right angle, is called **right-angled**.*

*The side opposite to the right angle is called the **hypotenuse**§.*



\* A figure, which has three sides, must also have three angles. It is for this reason called a **triangle**.

† Derived from *πολύς* "much" and *γωνία* "an angle".

‡ Derived from *ἴσος* "equal" and *σκέλος* "a leg".

§ Derived from *ὑπό* "under" and *τείνειν* "to stretch". ἡ ὑπο-  
τείνουσα γραμμή "the line subtending" or "stretching across" (the  
right angle).

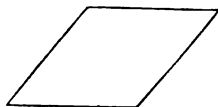
*A triangle, which has an obtuse angle, is called obtuse-angled.*



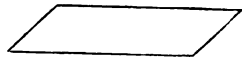
*A triangle, which has three acute angles, is called acute-angled.*



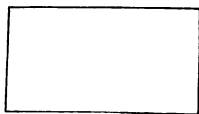
**DEFINITION 17.** *A quadrilateral, which has four sides equal, is called a rhombus.*



**DEFINITION 18.** *A quadrilateral, whose opposite sides are parallel, is called a parallelogram.*



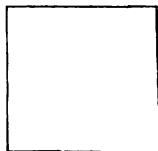
**DEFINITION 19.** *A parallelogram, one of whose angles is a right angle, is called a rectangle.*



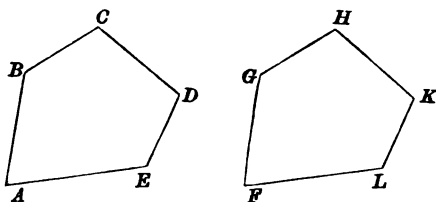
It will be proved later that each angle of a rectangle is a right angle.

**DEFINITION 20.** *A rectangle, which has two adjacent sides equal, is called a square.*

It will be proved later that all the sides of a square are equal.



**DEFINITION 21.** *Two figures are said to be equal in all respects, when it is possible to shift one unchanged in shape and size so as to coincide with the other.*

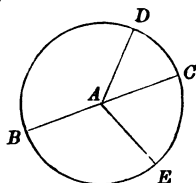


The figures  $ABCDE$ ,  $FGHLK$  are equal in all respects, if it be possible to shift  $ABCDE$  so that the vertices  $A, B, C, D, E$  may coincide with the vertices  $F, G, H, K, L$  respectively: in which case the sides of the two figures must be equal,  $AB, BC, CD, DE, EA$  to  $FG, GH, HK, KL, LF$  respectively, and the angles must be equal,  $ABC, BCD, CDE, DEA, EAB$  to  $FGH, GHK, HKL, KLF, LFG$  respectively.

**DEFINITION 22.** *A plane closed line, which is such that all straight lines drawn to it from a fixed point are equal, is called a circle.*

*This point is called the centre of the circle.*

It will be proved hereafter that a circle has only one centre.



*A straight line drawn from the centre of the circle to the circle is called a radius.*

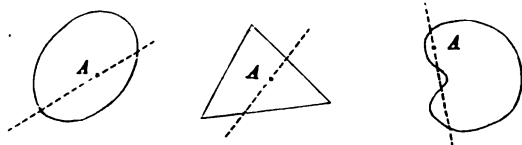
*A straight line drawn through the centre and terminated both ways by the circle is called a diameter.*

It will be proved hereafter that three points on a circle completely fix the position and magnitude of the circle: hence we generally denote a circle by mentioning three points on it; for instance the circle in the diagram might be called the circle  $BDE$ , or the circle  $DBC$ .

The one assumption which we make with reference to describing a circle is contained in the following postulate:

**POSTULATE 6.** *A circle may be described with any point as centre and with any straight line drawn from that point as radius.*

**POSTULATE 7.** *Any straight line drawn through a point within a closed figure must, if produced far enough, intersect the figure in two points at least.*



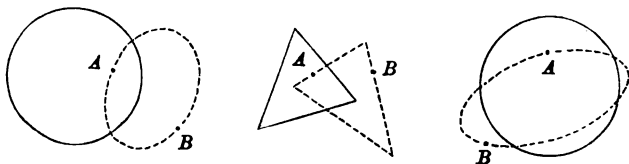
In the diagram we have three specimens of closed figures each with a point *A* inside the figure.

It is easily seen that any straight line through *A* must intersect the figure in two points at least: in the case of two of the figures a straight line cannot intersect the figure in more than two points: but in the third case, a straight line can be drawn to intersect the figure in four points.

**POSTULATE 8.** *Any line joining two points one within and the other without a closed figure must intersect the figure in one point at least.*

It follows that

*Any closed line drawn through two points one within and the other without a closed figure must intersect the figure in two points at least.*



In the diagram we have three specimens of closed figures with two points *A*, *B*, one inside and the other outside the figure.



It is easily seen that any line joining  $A$  and  $B$  must intersect the figure in one point at least, and that any closed line drawn through  $A$  and  $B$  must intersect the figure in two points at least: in two of the cases in the diagram either of the paths represented by part of the dotted line joining  $A$  and  $B$  intersects the figure in one point only and the closed line drawn intersects the figure in two points only: but in the third case one of the paths from  $A$  to  $B$  represented by part of the dotted line intersects the figure in one point only, while the path represented by the other part of the dotted line intersects the figure in three points, and the closed line drawn through  $A$  and  $B$  intersects the figure in four points.

**AXIOMS.** There are a number of simple propositions generally admitted to be true universally, i.e. with reference to magnitudes of all kinds.

Such propositions were called by Euclid κοινὰ ἔννοια, "common notions": they are now usually denominated *axioms*\*, ἀξιώματα, as being propositions claimed without demonstration.

The following are examples of such axioms:

*Things which are equal to the same thing are equal to one another.*

*If equals be added to equals, the wholes are equal.*

*If equals be taken from equals, the remainders are equal.*

*Doubles of equals are equal.*

*Halves of equals are equal.*

*The whole of a thing is greater than a part.*

*If one thing be greater than a second and the second greater than a third, the first is greater than the third.*

Such propositions as the above we shall use freely in the following pages without further remark.

\* Dr Johnson in his English Dictionary defined an axiom as "a proposition evident at first sight, that cannot be made plainer by demonstration."

## PROPOSITION 1.

*On a given finite straight line to construct an equilateral triangle.*

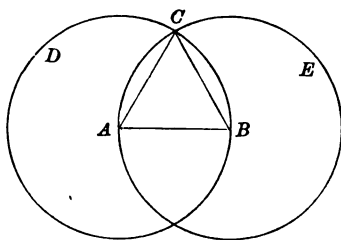
Let  $AB$  be the given finite straight line:  
it is required to construct an equilateral triangle on  $AB$ .

CONSTRUCTION. With  $A$  as centre and  $AB$  as radius,  
describe the circle  $BCD$ . (Post. 6.)

With  $B$  as centre and  $BA$  as radius, describe the circle  $ACE$ .

These circles must intersect: (Post. 8.)  
let them intersect in  $C$ .

Draw the straight lines  $CA$ ,  $CB$ : (Post. 3.)  
then  $ABC$  is a triangle constructed as required.



PROOF. Because  $A$  is the centre of the circle  $BCD$ ,  
 $AC$  is equal to  $AB$ . (Def. 22.)

And because  $B$  is the centre of the circle  $ACE$ ,  
 $BC$  is equal to  $BA$ .

Therefore  $CA$ ,  $AB$ ,  $BC$  are all equal.

Wherefore, the triangle  $ABC$  is equilateral, and it has  
been constructed on the given finite straight line  $AB$ .

It is assumed in this proposition that the two circles intersect. It is easily seen that they must intersect in *two* points. We can take either of these points as the third angular point of an equilateral triangle on the given straight line; there are thus *two* triangles which can be constructed satisfying the requirements of the proposition. We say therefore that the problem put before us in this proposition admits of two solutions.

We shall often have occasion to notice that a geometrical problem admits of more than one solution, and it is a very useful exercise to consider the number of possible solutions of a particular problem.

For the future we shall generally use the abbreviated expression "draw  $AB$ " instead of "draw the straight line  $AB$ " or "draw a straight line from the point  $A$  to the point  $B$ ."

#### EXERCISES.

1. Produce a straight line so as to be (a) twice, (b) three times, (c) five times, its original length.
2. Construct on a given straight line an isosceles triangle, such that each of its equal sides shall be (a) twice, (b) three times, (c) six times, the length of the given line.
3. Prove that, if two circles, whose centres are  $A, B$ , and whose radii are equal, intersect in  $C, D$ , the figure  $ABCD$  is a rhombus.

## PROPOSITION 2.

*From a given point to draw a straight line equal to a given straight line.*

Let  $A$  be the given point, and  $BC$  the given straight line: it is required to draw from  $A$  a straight line equal to  $BC$ .

CONSTRUCTION. Draw  $AB$ , the straight line from  $A$  to one of the extremities of  $BC$ ; (Post. 3.)

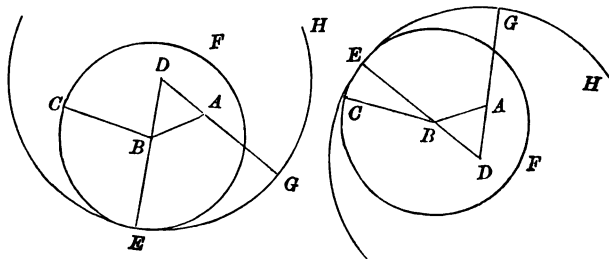
on it construct an equilateral triangle  $ABD$ . (Prop. 1.)

With  $B$  as centre and  $BC$  as radius, describe the circle  $CEF$ , (Post. 6.)

meeting  $DB$  (produced if necessary) at  $E$ . (Post. 7.)

With  $D$  as centre and  $DE$  as radius, describe the circle  $EGH$ , meeting  $DA$  (produced if necessary) at  $G$ :

then  $AG$  is a straight line drawn as required.



PROOF. Because  $B$  is the centre of the circle  $CEF$ ,  
 $BC$  is equal to  $BE$ . (Def. 22.)

Again, because  $D$  is the centre of the circle  $EGH$ ,

$DG$  is equal to  $DE$ ;

and because  $ABD$  is an equilateral triangle,

$DA$  is equal to  $DB$ ;

therefore  $AG$  is equal to  $BE$ . (Def. 14.)

And it has been proved that  $BC$  is equal to  $BE$ ;

therefore  $AG$  is equal to  $BC$ .

Wherefore, from the given point  $A$  a straight line  $AG$  has been drawn equal to the given straight line  $BC$ .

It is assumed in this proposition that the straight line  $DB$  intersects the circle  $CEF$ . It is easily seen that it must intersect it in two points.

It will be noticed that in the construction of this proposition there are several steps at which a choice of two alternatives is afforded: (1) we can draw either  $AB$  or  $AC$  as the straight line on which to construct an equilateral triangle: (2) we can construct an equilateral triangle on either side of  $AB$ : (3) if  $DB$  cut the circle in  $E$  and  $I$ , we can choose either  $DE$  or  $DI$  as the radius of the circle which we describe with  $D$  as centre.

There are therefore three steps in the construction, at each of which there is a choice of two alternatives: the total number of solutions of the problem is therefore  $2 \times 2 \times 2$  or eight.

On the opposite page two diagrams are drawn, to represent two out of these eight possible solutions. It will be a useful exercise for the student to draw diagrams corresponding to some of the remaining six.

### EXERCISES.

1. Draw a diagram for the case in which the given point is the middle point of the given straight line.
2. Draw a diagram for the case in which the given point is in the given straight line produced.
3. Draw from a given point a straight line (*a*) twice, (*b*) three times the length of a given straight line.
4. Draw from  $D$  in any one of the diagrams of Proposition 2 a straight line, so that the part of it intercepted between the two circles may be equal to the given straight line. Is a solution always possible?

## PROPOSITION 3.

*From the greater of two given straight lines to cut off a part equal to the less.*

Let  $AB$  and  $CD$  be the two given straight lines, of which  $AB$  is the greater :

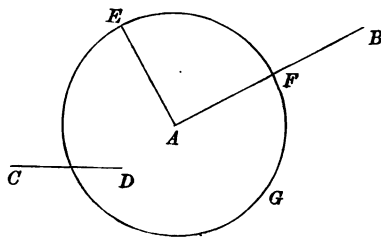
it is required to cut off from  $AB$  a part equal to  $CD$ .

CONSTRUCTION. From  $A$  draw a straight line  $AE$  equal to  $CD$ ; (Prop. 2.)

with  $A$  as centre and  $AE$  as radius,  
describe the circle  $EFG$ . (Post. 6.)

The circle must intersect  $AB$  between  $A$  and  $B$ ,  
for  $AB$  is greater than  $AE$ .

Let  $F$  be the point of intersection :  
then  $AF$  is the part required.



PROOF. Because  $A$  is the centre of the circle  $EFG$ ,  
 $AE$  is equal to  $AF$ . (Def. 22.)

But  $AE$  was made equal to  $CD$ ; (Construction.)  
therefore  $AF$  is equal to  $CD$ .

Wherefore, from  $AB$  the greater of two given straight lines a part  $AF$  has been cut off equal to  $CD$  the less.

The demand made in Postulate 6, that "a circle may be described with any point as centre and with any straight line drawn from that point as radius," is equivalent, in practical geometry, to saying that a pair of compasses may be used in the following manner: the extremity of one leg of a pair of compasses may be put down on any point *A*, the compasses may then be opened so that the extremity of the other leg comes to any other point and then a circle may be swept out by the extremity of the second leg of the compasses, the extremity of the first leg remaining throughout the motion on the point *A*.

Compasses are also used practically for carrying a given length from any one position to any other: for instance, they would generally be used to solve the problem of Proposition 3 by opening the compasses out till the extremities of the legs came to the points *C*, *D*: they would then be shifted, without any change in the opening of the legs, until the extremity of one leg was on *A* and the extremity of the other in the straight line *AB*.

Euclid restricted himself much in the same way as a draughtsman would, if he allowed himself only the first mentioned use of the compasses: the first three propositions shew how Euclid with this self-imposed restriction solved the problem, which without such a restriction could have been solved more readily.

After the problems in the first three propositions have been solved, we may assume that we can draw a circle, as a practical draughtsman would, with any point as centre and with a length equal to any given straight line as radius.

### EXERCISES.

1. On a given straight line describe an isosceles triangle having each of the equal sides equal to a second given straight line.
2. Construct upon a given straight line an isosceles triangle having each of the equal sides double of a second given straight line.
3. Construct a rhombus having a given angle for one of its angles, and having its sides each equal to a given straight line.

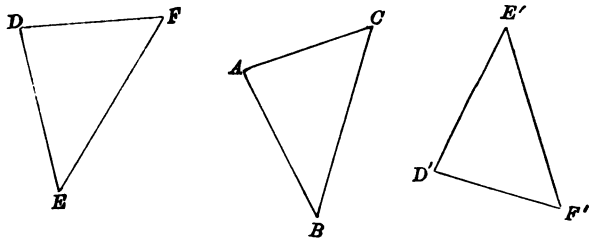
## PROPOSITION 4.

*If two triangles have two sides of the one equal to two sides of the other, and also the angles contained by those sides equal, the two triangles are equal in all respects.*

(See Def. 21.)

Let  $ABC$ ,  $DEF$  be two triangles, in which  $AB$  is equal to  $DE$ , and  $AC$  to  $DF$ , and the angle  $BAC$  is equal to the angle  $EDF$ :

it is required to prove that the triangles  $ABC$ ,  $DEF$  are equal in all respects.



PROOF. Because the angles  $BAC$ ,  $EDF$  are equal,  
 it is possible to shift the triangle  $ABC$   
 so that  $A$  coincides with  $D$ ,  
 and  $AB$  coincides in direction with  $DE$ ,  
 and  $AC$  with  $DF$ . (Test of Equality, page 8.)

If this be done,

because  $AB$  is equal to  $DE$ ,  
 $B$  must coincide with  $E$ ;  
 and because  $AC$  is equal to  $DF$ ,  
 $C$  must coincide with  $F$ .

Again because  $B$  coincides with  $E$  and  $C$  with  $F$ ,  
 $BC$  coincides with  $EF$ ; (Post. 2.)  
 therefore the triangle  $ABC$  coincides with the triangle  $DEF$ ,  
 and is equal to it in all respects.

Wherefore, *if two triangles &c.*



The proof of this proposition holds good not only for a pair of triangles such as  $ABC$ ,  $DEF$  in the diagram: it holds good equally for a pair such as  $ABC$ ,  $D'E'F'$ , one of which must be reversed or turned over before the triangles can be made to coincide or fit exactly.

In this proposition Euclid assumed Postulate 2, that two straight lines cannot have a common part. When the triangle  $ABC$  is shifted, so that  $A$  is on  $D$  and  $AB$  is on  $DE$ , there would be no justification for the conclusion that  $B$  must coincide with  $E$ , because  $AB$  is equal to  $DE$ , if it were possible for two straight lines to have a common part. In fact, two curved lines might be drawn from the point  $D$  starting in the same direction  $DE$  but leading to two totally distinct points  $E$  and  $F$  although the lines were of the same length. It is tacitly assumed, that if the lines be straight lines, this is impossible.

#### EXERCISES.

1. If the straight line joining the middle points of two opposite sides of a quadrilateral be at right angles to each of these sides, the other two sides are equal.

2. If in a quadrilateral  $ABCD$  the sides  $AB$ ,  $CD$  be equal and the angles  $ABC$ ,  $BCD$  be equal, the diagonals  $AC$ ,  $BD$  are equal.

3. If in a quadrilateral two opposite sides be equal, and the angles which a third side makes with the equal sides be equal, the other angles are equal.

4. Prove by the method of superposition that, if in two quadrilaterals  $ABCD$ ,  $A'B'C'D'$ , the sides  $AB$ ,  $BC$ ,  $CD$  be equal to the sides  $A'B'$ ,  $B'C'$ ,  $C'D'$  respectively, and the angles  $ABC$ ,  $BCD$  equal to the angles  $A'B'C'$ ,  $B'C'D'$  respectively, the quadrilaterals are equal in all respects.

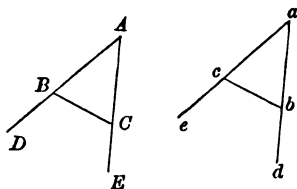
## PROPOSITION 5.

*If two sides of a triangle be equal, the angles opposite to these sides are equal, and the angles made by producing these sides beyond the third side are equal.*

Let  $ABC$  be a triangle, in which  $AB$  is equal to  $AC$ , and  $AB, AC$  are produced to  $D, E$ :

it is required to prove that the angle  $ACB$  is equal to the angle  $ABC$ , and the angle  $BCE$  to the angle  $CBD$ .

CONSTRUCTION. Let the figure  $ABCDE$  be turned over and shifted unchanged in shape and size to the position  $abcde$ ,  $A$  to  $a$ ,  $B$  to  $b$ ,  $C$  to  $c$ ,  $D$  to  $d$  and  $E$  to  $e$ .



PROOF. Because the angles  $DAE, ead$  are equal, it is possible to shift the figure  $abcde$

so that  $a$  coincides with  $A$ ,

and  $ae$  coincides in direction with  $AD$ ,

and  $ad$  with  $AE$ . (Test of Equality, page 8.)

If this be done,

because  $ac$  is equal to  $AB$ ,

$c$  must coincide with  $B$ ;

and because  $ab$  is equal to  $AC$ ,

$b$  must coincide with  $C$ ;

hence  $cb$  coincides with  $BC$ . (Post. 2.)

Now because  $ace$  coincides in direction with  $ABD$ ,

and  $cb$  with  $BC$ ,

the angle  $acb$  coincides with the angle  $ABC$ ,

and the angle  $bce$  with the angle  $CBD$ ;

therefore the angle  $acb$  is equal to the angle  $ABC$ ,

and the angle  $bce$  to the angle  $CBD$ .

But the angle  $acb$  is equal to the angle  $ACB$ ,  
and the angle  $bce$  to the angle  $BCE$ ;  
therefore the angle  $ACB$  is equal to the angle  $ABC$ ,  
and the angle  $BCE$  to the angle  $CBD$ .

Wherefore, if two sides &c.

COROLLARY 1. *An equilateral triangle is also equiangular.*

COROLLARY 2. *If two angles of a triangle be unequal, the sides opposite to these angles are unequal.*

#### EXERCISES.

1. The opposite angles of a rhombus are equal.
2. If a quadrilateral have two pairs of equal adjacent sides, it has one pair of opposite angles equal.
3. If in a quadrilateral  $ABCD$ ,  $AB$  be equal to  $AD$  and  $BC$  to  $DC$ , the diagonal  $AC$  bisects each of the angles  $BAD$ ,  $BCD$ .
4. If in a quadrilateral  $ABCD$ ,  $AB$  be equal to  $AD$  and  $BC$  to  $DC$ , the diagonal  $BD$  is bisected at right angles by the diagonal  $AC$ .
5. Prove that the triangle, whose vertices are the middle points of the sides of an equilateral triangle, is equilateral.
6. Prove that the triangle, formed by joining the middle points of the sides of an isosceles triangle, is isosceles.
7. Prove by the method of superposition that, if in a convex quadrilateral  $ABCD$ ,  $AB$  be equal to  $CD$  and the angle  $ABC$  to the angle  $BCD$ ,  $AD$  is parallel to  $BC$ .

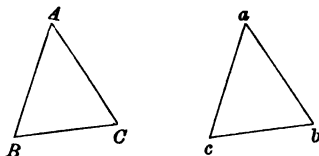
## PROPOSITION 6.

*If two angles of a triangle be equal, the sides opposite to these angles are equal.*

Let  $ABC$  be a triangle, in which the angle  $ABC$  is equal to the angle  $ACB$ :

it is required to prove that  $AC$  is equal to  $AB$ .

CONSTRUCTION. Let the triangle  $ABC$  be turned over and shifted unchanged in shape and size to the position  $abc$ ,  $A$  to  $a$ ,  $B$  to  $b$ , and  $C$  to  $c$ .



PROOF. Because the sides  $BC$ ,  $cb$  are equal,  
 it is possible to shift the triangle  $acb$   
 so that  $cb$  coincides with  $BC$ ,  $c$  with  $B$ , and  $b$  with  $C$ ,  
 (Test of Equality, page 5.)  
 and so that the triangles  $acb$ ,  $ABC$  are on the same side  
 of  $BC$ .

If this be done,

because the angles  $ABC$ ,  $acb$  are equal,

$ca$  must coincide in direction with  $BA$ ;

and because the angles  $ACB$ ,  $abc$  are equal,

$ba$  must coincide in direction with  $CA$ .

And because two straight lines cannot intersect in more  
 than one point, (Post. 1.)

the point  $a$ , which is the intersection of  $ca$  and  $ba$ , must  
 coincide with the point  $A$ , which is the intersection of  
 $BA$  and  $CA$ .

Now because  $a$  coincides with  $A$  and  $c$  with  $B$ ,  
 $ac$  coincides with  $AB$  and is equal to it.

But  $ac$  is equal to  $AC$ ;

therefore  $AC$  is equal to  $AB$ .

Wherefore *if two angles &c.*

COROLLARY. *An equiangular triangle is also equilateral.*

When in two propositions the hypothesis of each is the conclusion of the other, each proposition is said to be the **converse** of the other.

The theorems in Propositions 5 and 6 are the converses of each other.

It must not be assumed that the converse of a proposition is necessarily true.

## EXERCISES.

1. Shew that, if the angles  $ABC$  and  $ACB$  at the base of an isosceles triangle be bisected by the straight lines  $BD$  and  $CD$ ,  $DBC$  will be an isosceles triangle.

2.  $BAC$  is a triangle having the angle  $B$  double of the angle  $A$ . If  $BD$  bisect the angle  $B$  and meet  $AC$  at  $D$ ,  $BD$  is equal to  $AD$ .

3. Prove by the method of superposition that, if in two triangles  $ABC$ ,  $A'B'C'$  the angles  $ABC$ ,  $BCA$  be equal to the angles  $A'B'C'$ ,  $B'C'A'$  respectively and the sides  $BC$ ,  $B'C'$  be equal, the triangles are equal in all respects.

4. Prove by the method of superposition that, if in two quadrilaterals  $ABCD$ ,  $A'B'C'D'$  the angles  $DAB$ ,  $ABC$ ,  $BCD$  be equal to the angles  $D'A'B'$ ,  $A'B'C'$ ,  $B'C'D'$  respectively, and the sides  $AB$ ,  $BC$  be equal to the sides  $A'B'$ ,  $B'C'$  respectively, the quadrilaterals are equal in all respects.

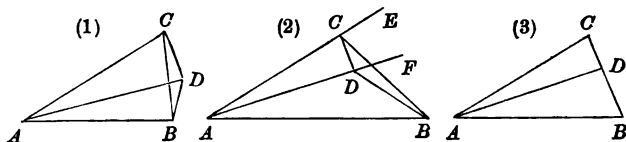
5. If in a quadrilateral  $ABCD$ ,  $AB$  be equal to  $AD$  and the angle  $ABC$  to the angle  $ADC$ , then  $BC$  is equal to  $DC$ , and the diagonal  $AC$  bisects the quadrilateral and two of its angles.

## PROPOSITION 7.

*If two points on the same side of a straight line be equidistant from one point in the line, they cannot be equidistant from any other point in the line.*

Let  $AB$  be a given straight line, and  $C, D$  be two points on the same side of it equidistant from the point  $A$ : it is required to prove that  $C, D$  cannot be equidistant from any other point in the line.

CONSTRUCTION. Take any other point  $B$  in the line, and draw  $BC, BD$ .



- PROOF. Because  $C$  and  $D$  are two different points, either
- (1) the vertex of each of the triangles  $ABC, ABD$  must be outside the other triangle,
  - (2) the vertex of one triangle must be inside the other,
  - or (3) the vertex of one triangle must be on a side of the other.
- (1) First let the vertex of each triangle be without the other.

Because  $AD$  is equal to  $AC$ ,  
the angle  $ACD$  is equal to the angle  $ADC$ . (Prop. 5.)

But the angle  $ACD$  is greater than the angle  $BCD$ ,  
and the angle  $BDC$  is greater than the angle  $ADC$ :  
therefore the angle  $BDC$  is greater than the angle  $BCD$ ;  
therefore  $BC, BD$  are unequal. (Prop. 5, Coroll. 2.)

- (2) Next let the vertex  $D$  of one triangle  $ABD$  be within the other triangle  $ABC$ :  
produce  $AC, AD$  to  $E, F$ . (Post. 4.)

Then because in the triangle  $ACD$ ,  $AC$  is equal to  $AD$ , the angles  $ECD$ ,  $FDC$  made by producing the sides  $AC$ ,  $AD$  are equal. (Prop. 5.)

But the angle  $ECD$  is greater than the angle  $BCD$ , and the angle  $BDC$  is greater than the angle  $FDC$ ; therefore the angle  $BDC$  is greater than the angle  $BCD$ ; therefore  $BC$ ,  $BD$  are unequal. (Prop. 5, Coroll. 2.)

(3) Next let the vertex  $D$  of one triangle lie on one of the sides  $BC$  of the other :  
then  $BC$ ,  $BD$  are unequal.

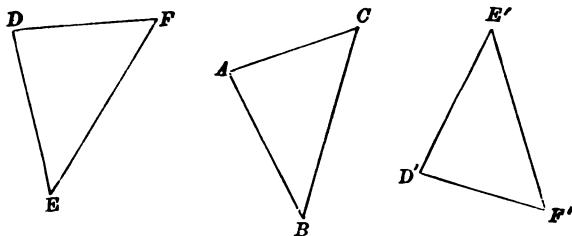
Wherefore, *if two points on the same side &c.*

## PROPOSITION 8.

*If two triangles have three sides of the one equal to three sides of the other, the triangles are equal in all respects.*

Let  $ABC$ ,  $DEF$  be two triangles, in which  $AB$  is equal to  $DE$ ,  $AC$  to  $DF$ , and  $BC$  to  $EF$ :

it is required to prove that the triangles  $ABC$ ,  $DEF$  are equal in all respects.



**PROOF.** Because the sides  $BC$ ,  $EF$  are equal,  
it is possible to shift the triangle  $ABC$ ,  
so that  $BC$  coincides with  $EF$ ,  $B$  with  $E$  and  $C$  with  $F$ ,  
(Test of Equality, page 5.)  
and so that the triangles  $ABC$ ,  $DEF$  are on the same side  
of  $EF$ .

If this be done,

$A$  must coincide with  $D$ :

for there cannot be two points on the same side of the  
straight line  $EF$  equidistant from  $E$   
and also equidistant from  $F$ . (Prop. 7.)

Now because  $A$  coincides with  $D$ ,

and  $B$  coincides with  $E$ , (Constr.)

$AB$  coincides with  $DE$ . (Post. 2.)

Similarly it can be proved that

$AC$  coincides with  $DF$ .

Therefore the triangle  $ABC$  coincides with the triangle  $DEF$ ,  
and is equal to it in all respects.

Wherefore, *if two triangles &c.*



## EXERCISES.

1. If a quadrilateral have two pairs of equal sides, it must have one pair and may have two pairs of equal angles.

2.  $ABC$ ,  $DBC$  are two isosceles triangles on the same base  $BC$ , and on the same side of it: shew that  $AD$  bisects the vertical angles of the triangles.

3. If the opposite sides of a quadrilateral be equal, the opposite angles are equal.

4. If in a quadrilateral two opposite sides be equal, and the diagonals be equal, the quadrilateral has two pairs of equal angles.

5. If in a quadrilateral the sides  $AB$ ,  $CD$  be equal and the angles  $ABC$ ,  $BCD$  be equal, the angles  $CDA$ ,  $DAB$  are equal.

6. The sides  $AB$ ,  $AD$  of a quadrilateral  $ABCD$  are equal, and the diagonal  $AC$  bisects the angle  $BAD$ ; shew that the sides  $CB$ ,  $CD$  are equal, and that the diagonal  $AC$  bisects the angle  $BCD$ .

7.  $ACB$ ,  $ADB$  are two triangles on the same side of  $AB$ , such that  $AC$  is equal to  $BD$ , and  $AD$  is equal to  $BC$ , and  $AD$  and  $BC$  intersect at  $O$ : shew that the triangle  $AOB$  is isosceles.

8. A diagonal of a rhombus bisects each of the angles through which it passes.

## PROPOSITION 9.

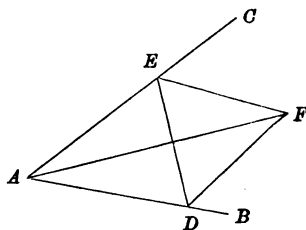
*To bisect a given angle.*

Let  $BAC$  be the given angle:  
it is required to bisect it.

CONSTRUCTION. Take any point  $D$  in  $AB$ ,  
and from  $AC$  cut off  $AE$  equal to  $AD$ . (Prop. 3.)

Draw  $DE$ ,  
and on  $DE$ , on the side away from  $A$ , construct the equilateral triangle  $DEF$ . (Prop. 1.)

Draw  $AF$ :  
then  $AF$  is the required bisector of the angle  $BAC$ .



PROOF. Because in the triangles  $DAF$ ,  $EAF$ ,  
 $DA$  is equal to  $EA$ ,  
 $AF$  to  $AF$ ,  
and  $FD$  to  $FE$ ,

the triangles are equal in all respects; (Prop. 8.)  
therefore the angle  $DAF$  is equal to the angle  $EAF$ .

Wherefore, the given angle  $BAC$  is bisected by the straight line  $AF$ .

In the construction for this proposition it is said that the equilateral triangle  $DEF$  is to be constructed on the side of  $DE$  away from  $A$ . This restriction is introduced in order to prevent the possibility of the point  $F$  coinciding with the point  $A$ .

In practical geometry it is always desirable to obtain two points, which determine a straight line, as far apart as possible, as then an error in the position of one of the points causes less error in the position of the straight line.

## EXERCISES.

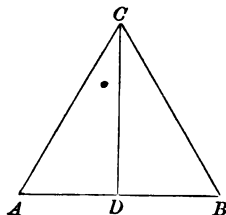
1. A straight line, bisecting the angle contained by two equal sides of a triangle, bisects the third side.
2. The bisectors of the angles  $ABC$ ,  $ACB$  of a triangle  $ABC$  meet in  $D$ : prove that, if  $DB$ ,  $DC$  be equal,  $AB$ ,  $AC$  are equal.
3. Prove that there is only one bisector of a given angle.
4. Prove that the bisectors of the angles of an equilateral triangle meet in a point.
5. Prove that the bisectors of the angles of an isosceles triangle meet in a point.
6.  $BAC$  is a given angle; cut off  $AB$ ,  $AC$  equal to one another: with centres  $B$ ,  $C$  describe circles having equal radii: if the circles intersect at  $D$ ,  $AD$  bisects the angle  $BAC$ .
7. Prove by the method of superposition that, if one diagonal of a quadrilateral bisect each of the angles through which it passes, the two diagonals are at right angles to each other.

## PROPOSITION 10.

*To bisect a given finite straight line.*

Let  $AB$  be the given finite straight line:  
it is required to bisect it.

CONSTRUCTION. On  $AB$  construct an equilateral triangle  $ABC$ , (Prop. 1.)  
and bisect the angle  $ACB$  by the straight line  $CD$ , meeting  $AB$  at  $D$ : (Prop. 9.)  
then  $AB$  is bisected as required at  $D$ .



PROOF. Because in the triangles  $ACD$ ,  $BCD$ ,  
 $AC$  is equal to  $BC$ ,  
and  $CD$  to  $CD$ ,  
and the angle  $ACD$  is equal to the angle  $BCD$ ,  
the triangles are equal in all respects; (Prop. 4.)  
therefore  $AD$  is equal to  $BD$ .

Wherefore, *the given finite straight line  $AB$  is bisected at the point  $D$ .*

## EXERCISES.

1. Prove that there is only one point of bisection of a given finite straight line.
2. If two circles intersect, then the straight line joining their centres bisects at right angles the straight line joining their points of intersection.
3. Draw from the vertex of a triangle to the opposite side a straight line, which shall exceed the smaller of the other sides as much as it is exceeded by the greater.

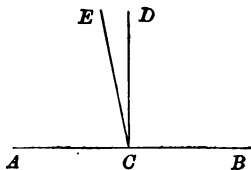
## PROPOSITION 10 A.

*From the same point in a given straight line and on the same side of it, only one straight line can be drawn at right angles to the given straight line.*

From the point  $C$  in the straight line  $AB$  let the straight line  $CD$  be drawn at right angles to  $AB$ :

it is required to prove that no other straight line can be drawn from  $C$  at right angles to  $AB$  on the same side of it.

CONSTRUCTION. Draw from  $C$  within the angle  $ACD$  any straight line  $CE$ .



PROOF. Because the angles  $DCB$ ,  $DCA$  are equal,  
 and the angle  $ECB$  is greater than the angle  $DCB$ ,  
 and the angle  $DCA$  is greater than the angle  $ECA$ ;  
 therefore the angle  $ECB$  is greater than the angle  $ECA$ ;  
 therefore  $CE$  is not at right angles to  $AB$ . (Def. 11.)  
 Similarly it can be proved that no straight line drawn  
 from  $C$  within the angle  $DCB$  can be at right angles  
 to  $AB$ .

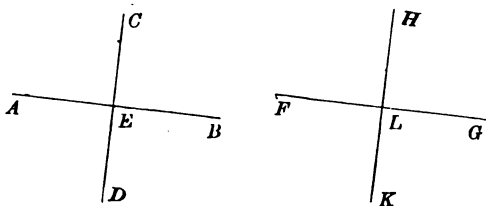
Therefore no straight line other than  $CD$  drawn from  $C$   
 can be at right angles to  $AB$  on the same side of it.

Wherefore, *from the same point &c.*

## PROPOSITION 10 B.

*All right angles are equal to one another.*

Let the straight lines  $AB$ ,  $CD$  meet at  $E$  and make the angles  $CEA$ ,  $CEB$  right angles; (Def. 11.)  
and let the straight lines  $FG$ ,  $HK$  meet at  $L$  and make the angles  $HLF$ ,  $HLG$  right angles :  
it is required to prove that the angle  $CEA$  is equal to the angle  $HLF$ .



PROOF. If the figure  $ABCDE$  be shifted, so that  $E$  coincides with  $L$ , and the line  $AB$  in direction with  $FG$ , and so that  $EC$ ,  $LH$  are on the same side of  $FG$  :

then  $EC$  must coincide with  $LH$ , for at the same point  $L$  in  $FG$  on the same side of it there cannot be two straight lines at right angles to  $FG$ ; (Prop. 10 A.)

therefore the angle  $AEC$  coincides with the angle  $FLH$ ,  
and is equal to it.

Wherefore, *all right angles &c.*

## PROPOSITION 11.

*To draw a straight line at right angles to a given straight line from a given point in it.*

Let  $AB$  be the given straight line, and  $C$  the given point in it :

it is required to draw from  $C$  a straight line at right angles to  $AB$ .

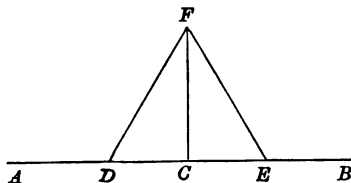
CONSTRUCTION. Take any point  $D$  in  $AC$ ,

and from  $CB$  cut off  $CE$  equal to  $CD$ . (Prop. 3.)

On  $DE$  construct an equilateral triangle  $DFE$ , (Prop. 1.)

and draw  $CF$  :

then  $CF$  is a straight line drawn as required.



PROOF. Because in the triangles  $DCF$ ,  $ECF$ ,

$DC$  is equal to  $EC$ ,

$CF$  to  $CF$ ,

and  $FD$  to  $FE$ ,

(Constr.)

the triangles are equal in all respects ;

(Prop. 8.)

therefore the angle  $DCF$  is equal to the angle  $ECF$ ,

and they are adjacent angles ;

therefore each of these angles is a right angle ; and the straight lines are at right angles to each other. (Def. 11.)

Wherefore,  $CF$  has been drawn at right angles to the given straight line  $AB$ , from the given point  $C$  in it.



In the definition of a circle (Def. 22) we meet with the idea of a point, which moves subject to a given condition, the condition being that the point is always to be at a given distance from a given point, i.e. from the centre of the circle. The *path* of such a moving point, or the place (*locus*), at some position on which the point must always be and at any position on which the point may be, is called the *locus* of the point. Hence we say in the case just mentioned that the locus of a point which is at a given distance from a given point is a circle.

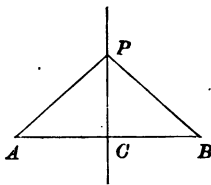
As a further illustration of this idea, let us consider the locus of a point, which moves subject to the condition that it is always to be equidistant from two given points.

Let  $A, B$  be two given points, and let  $P$  be a point equidistant from  $A$  and  $B$ , i.e. let  $PA$  be equal to  $PB$ .

Draw  $AB$  and take  $C$  the middle point of  $AB$ .

Draw  $PC$ .

Then because in the triangles  $PCA$ ,  $PCB$ ,



$PA$  is equal to  $PB$ ,  $PC$  to  $PC$ , and  $AC$  to  $BC$ ,

the two triangles are equal in all respects: (Prop. 8.)

therefore the angle  $PCA$  is equal to the angle  $PCB$ ,

and since they are adjacent angles each is a right angle.

It follows that, if  $P$  be equidistant from  $A$  and  $B$ , it must lie on the straight line  $CP$  which bisects  $AB$  at right angles. Every point on  $CP$  satisfies this condition.

We may state the result of this proposition thus: *The locus of a point equidistant from two given points is the straight line which bisects at right angles the straight line joining the given points.*

### EXERCISES.

1. The diagonals of a rhombus bisect each other at right angles.
2. Find in a given straight line a point equidistant from two given points. Is a solution always possible?
3. Find a point equidistant from three given points.
4. In the base  $BC$  of a triangle  $ABC$  any point  $D$  is taken. Draw a straight line such that, if the triangle  $ABC$  be folded along this straight line, the point  $A$  shall fall upon the point  $D$ .

## PROPOSITION 12.

*To draw a straight line at right angles to a given straight line from a given point without it.*

Let  $AB$  be the given straight line, and  $C$  the given point without it:

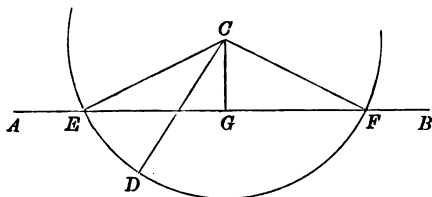
it is required to draw from  $C$  a straight line at right angles to  $AB$ .

CONSTRUCTION. Take any point  $D$  on the side of  $AB$  away from  $C$ , draw  $CD$ , and with  $C$  as centre and  $CD$  as radius, describe the circle  $DEF$ , (Post. 6.)

meeting  $AB$  (produced if necessary) at  $E$  and  $F$ .

Draw  $CE$ ,  $CF$ , and bisect the angle  $ECF$  by the straight line  $CG$  meeting  $AB$  at  $G$ : (Prop. 9.)

then  $CG$  is a straight line drawn as required.



PROOF. Because in the triangles  $ECG$ ,  $FCG$ ,

$EC$  is equal to  $FC$ ,

and  $CG$  to  $CG$ ,

and the angle  $ECG$  is equal to the angle  $FCG$ ,

the triangles are equal in all respects; (Prop. 4.)

therefore the angle  $CGE$  is equal to the angle  $CGF$ ,

and they are adjacent angles.

Therefore the straight lines  $CG$ ,  $AB$  are at right angles to each other. (Def. 11.)

Wherefore,  $CG$  has been drawn at right angles to the given straight line  $AB$  from the given point  $C$  without it.

In the construction for this proposition it is said that the point  $D$  is to be taken on the side of  $AB$  away from  $C$ . This restriction is introduced as a means of ensuring the intersection of the circle  $DEF$  with the straight line  $AB$ .

## EXERCISES.

1. Through two given points on opposite sides of a given straight line draw two straight lines, which shall meet in the given line and include an angle bisected by that line. In what case can there be more than one solution?

2. From two given points on the same side of a given straight line, draw two straight lines, which shall meet at a point in the given line and make equal angles with it.

3. Prove by the method of superposition that, if the perpendiculars on a given straight line from two points on the same side of it be equal, the straight line joining the points is parallel to the given line.

## PROPOSITION 13.

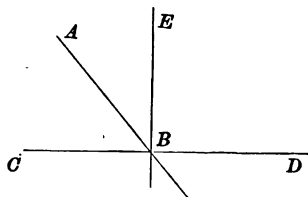
*The sum of the angles, which one straight line makes with another straight line on one side of it, is equal to two right angles.*

Let the straight line  $AB$  make with the straight line  $CD$ , on one side of it, the angles  $ABC$ ,  $ABD$  :

it is required to prove that the sum of these angles is equal to two right angles.

If the angle  $ABC$  be equal to the angle  $ABD$ ,  
each of them is a right angle,  
and their sum is equal to two right angles.

CONSTRUCTION. If the angles  $ABC$ ,  $ABD$  be not equal, from the point  $B$  draw  $BE$  at right angles to  $CD$ ; (Prop. 11.)  
 $BE$  cannot coincide with  $BA$ ;  
let it lie within the angle  $ABD$ .



PROOF. Now the angle  $CBE$  is the sum of the angles  $CBA$ ,  $ABE$ ;

to each of these equals add the angle  $EBD$ ;  
then the sum of the angles  $CBE$ ,  $EBD$  is equal to the sum of the angles  $CBA$ ,  $ABE$ ,  $EBD$ .

Again, the angle  $DBA$  is equal to the sum of the angles  $DBE$ ,  $EBA$ ;

to each of these equals add the angle  $ABC$ ;  
then the sum of the angles  $DBA$ ,  $ABC$  is equal to the sum of the angles  $DBE$ ,  $EBA$ ,  $ABC$ .

And the sum of the angles  $CBE$ ,  $EBD$  has been proved to be equal to the sum of the same three angles.

Therefore the sum of the angles  $CBE$ ,  $EBD$  is equal to the sum of the angles  $DBA$ ,  $ABC$ .

But  $CBE$ ,  $EBD$  are two right angles; (Constr.) therefore the sum of the angles  $DBA$ ,  $ABC$  is equal to two right angles.

Wherefore, *the sum of the angles &c.*

**COROLLARY.** *The sum of the four angles, which two intersecting straight lines make with one another, is equal to four right angles.*

#### EXERCISES.

1. Prove in the manner of Proposition 13 that, if  $A$ ,  $B$ ,  $C$ ,  $D$  be four points in order on a straight line, the sum of  $AB$ ,  $BD$  is equal to the sum of  $AC$ ,  $CD$ .

2. If one of the four angles, which two intersecting straight lines make with one another, be a right angle, all the others are right angles.

3. Prove by the method of superposition that only one perpendicular can be drawn to a given straight line from a given point without it.

4. Prove by the method of superposition that, if two right-angled triangles have their hypotenuses equal and two other angles equal, the triangles are equal in all respects.

5. A given angle  $BAC$  is bisected; if  $CA$  be produced to  $G$  and the angle  $BAG$  bisected, the two bisecting lines are at right angles.

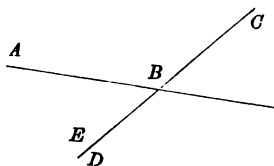
## PROPOSITION 14.

*If, at a point in a straight line, two other straight lines, on opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines are in the same straight line.*

At the point  $B$  in the straight line  $AB$ , let the two straight lines  $BC$ ,  $BD$ , on opposite sides of  $AB$ , make the adjacent angles  $ABC$ ,  $ABD$  together equal to two right angles:

it is required to prove that  $BD$  is in the same straight line with  $CB$ .

CONSTRUCTION. Produce  $CB$  to  $E$ .



PROOF. Because the straight line  $AB$  makes with the straight line  $CBE$ , on one side of it, the angles  $ABC$ ,  $ABE$ , the sum of these angles is equal to two right angles.

(Prop. 13.)

But the sum of the angles  $ABC$ ,  $ABD$  is equal to two right angles.

Therefore the sum of the angles  $ABC$ ,  $ABD$  is equal to the sum of the angles  $ABC$ ,  $ABE$ .

From each of these equals take away the angle  $ABC$ ;

then the angle  $ABD$  is equal to the angle  $ABE$ ;

therefore the line  $BD$  coincides in direction with  $BE$ ,

and is in the same straight line with  $CB$ .

Wherefore, *if at a point &c.*

**DEFINITION.** *Two angles, which are together equal to two right angles, are called **supplementary angles**, and each angle is said to be the **supplement** of the other.*

*Two angles, which are together equal to one right angle, are called **complementary angles**, and each angle is said to be the **complement** of the other.*

### EXERCISES.

1. If  $E$  be the middle point of the diagonal  $AC$  of a quadrilateral  $ABCD$ , whose opposite sides are equal,  $B, E, D$  lie on a straight line.

2. If  $OA, OB, OC, OD$  be four straight lines drawn in order from  $O$ , such that the angles  $BOC, DOA$  are equal and also the angles  $AOB, COD$ , then the lines  $OA, OC$  are in the same straight line and also the lines  $OB, OD$ .

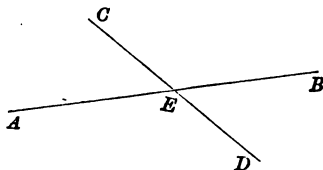
3. If it be possible within a quadrilateral  $ABCD$ , whose opposite sides are equal, to find a point  $E$  such that  $EA, EC$  are equal, and  $EB, ED$  are equal, then  $AEC, BED$  are straight lines.

4. If it be possible within a quadrilateral  $ABCD$ , whose opposite sides are equal, to find a point  $E$ , such that  $EA, EB, EC, ED$  are equal, then the quadrilateral is equiangular.

## PROPOSITION 15.

*If two straight lines intersect, vertically opposite angles are equal.*

Let the two straight lines  $AB$ ,  $CD$  intersect at  $E$ :  
it is required to prove that the angle  $AEC$  is equal to the angle  $DEB$ , and the angle  $CEB$  to the angle  $AED$ .



PROOF. The sum of the angles  $CEA$ ,  $AED$ , which  $AE$  makes with  $CD$  on one side of it, is equal to two right angles. (Prop. 13.)

Again, the sum of the angles  $AED$ ,  $DEB$ , which  $DE$  makes with  $AB$  on one side of it, is equal to two right angles.

Therefore the sum of the angles  $CEA$ ,  $AED$  is equal to the sum of the angles  $AED$ ,  $DEB$ .

From each of these equals take away the common angle  $AED$ ;

then the angle  $CEA$  is equal to the angle  $DEB$ .

Similarly it may be proved that the angle  $CEB$  is equal to the angle  $AED$ .

Wherefore, *if two straight lines &c.*



## EXERCISES.

1. If the diagonals of a quadrilateral bisect one another, opposite sides are equal.

2. In a given straight line find a point such that the straight lines, joining it to each of two given points on the same side of the line, make equal angles with it.

3.  $A, B$  are two given points;  $CD, DE$  two given straight lines: find points  $P, Q$  in  $CD, DE$ , such that  $AP, PQ$  are equally inclined to  $CD$ , and  $PQ, QB$  equally inclined to  $DE$ .

4. A straight line is drawn terminated by one of the sides of an isosceles triangle, and by the other side produced, and bisected by the base: prove that the straight lines thus intercepted between the vertex of the isosceles triangle, and this straight line, are together equal to the two equal sides of the triangle.

## PROPOSITION 16.

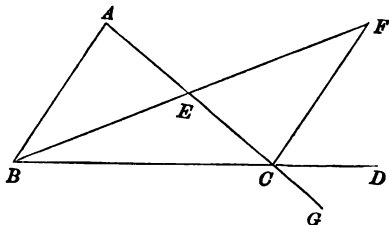
*An exterior angle of a triangle is greater than either of the interior opposite angles.*

Let  $ABC$  be a triangle, and let  $ACD$  be the exterior angle made by producing the side  $BC$  to  $D$ :

it is required to prove that the angle  $ACD$  is greater than either of the interior opposite angles  $CBA$ ,  $BAC$ .

CONSTRUCTION. Bisect  $AC$  at  $E$ . (Prop. 10.)

Draw  $BE$  and produce it to  $F$ ,  
making  $EF$  equal to  $EB$ , (Prop. 3.)  
and draw  $FC$ .



PROOF. Because in the triangles  $AEB$ ,  $CEF$ ,

$AE$  is equal to  $CE$ ,

and  $EB$  to  $EF$ ,

and the angle  $AEB$  is equal to the angle  $CEF$ ,

the triangles are equal in all respects; (Prop. 4.)  
therefore the angle  $BAE$  (or  $BAC$ ) is equal to the angle  $FCE$ .

Now the angle  $ECD$  (or  $ACD$ ) is greater than the angle  $ECF$ .

Therefore the angle  $ACD$  is greater than the angle  $BAC$ .

Similarly it can be proved that the angle  $BCG$ , which is made by producing  $AC$  and is equal to the angle  $ACD$ , is greater than the angle  $ABC$ .

Wherefore, *an exterior angle &c.*

## EXERCISES.

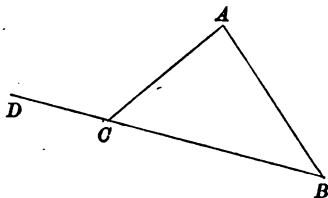
1. Only one perpendicular can be drawn to a given straight line from a given point without it.
2. Shew by joining the angular point  $A$  of a triangle to any point in the opposite side  $BC$  between  $B$  and  $C$  that the angles  $ABC$ ,  $BCA$  are together less than two right angles.
3. Not more than two equal straight lines can be drawn from a given point to a given straight line.
4. Prove by the method of superposition that, if a quadrilateral be equiangular, its opposite sides are equal.
5. Prove by the method of superposition that two right-angled triangles, which have their hypotenuses equal and one side equal to one side, are equal in all respects.

## PROPOSITION 17.

*The sum of any two angles of a triangle is less than two right angles.*

Let  $ABC$  be a triangle :  
it is required to prove that the sum of any two of its angles  
is less than two right angles.

CONSTRUCTION. Produce any side  $BC$  to  $D$ .



PROOF. Because  $ACD$  is an exterior angle of the triangle  $ABC$ ,

it is greater than the interior opposite angle  $ABC$ .

(Prop. 16.)

To each of these unequals add the angle  $ACB$  :

then the sum of the angles  $ACD$ ,  $ACB$  is greater than the sum of the angles  $ABC$ ,  $ACB$ .

But the sum of the angles  $ACD$ ,  $ACB$  is equal to two right angles.

(Prop. 13.)

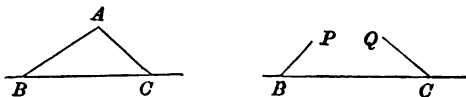
Therefore the sum of the angles  $ABC$ ,  $ACB$  is less than two right angles.

Similarly it can be proved that the sum of the angles  $BAC$ ,  $ACB$  is less than two right angles;

and also the sum of the angles  $CAB$ ,  $ABC$ .

Wherefore, *the sum of any two angles &c.*

The theorem established in this proposition may be stated thus: If from two points  $B, C$  in the straight line  $BC$  two straight lines be drawn which meet at any point  $A$ , then the sum of the angles  $ABC, ACB$  is less than two right angles.



We shall assume as a postulate the converse of this theorem, which may be stated thus

If from two points  $B, C$  in the straight line  $BC$  two straight lines  $BP, CQ$  be drawn making the sum of the angles  $PBC, QCB$  on the same side of  $BC$  less than two right angles, the two lines  $BP, CQ$  will meet if produced far enough.

It may be observed that the theorem established in Proposition 17 proves that the lines  $PB, QC$  cannot meet when produced beyond  $B$  and  $C$ : if therefore the postulate just stated be allowed, it follows that the lines  $BP, CQ$  must meet when produced beyond  $P$  and  $Q$ .

The postulate which we here assume may be stated in general terms as follows

**POSTULATE 9.** *If the sum of the two interior angles, which two straight lines make with a given straight line on the same side of it, be not equal to two right angles, the two straight lines are not parallel.*

### EXERCISES.

1. A triangle must have at least two acute angles.
2. Assuming Postulate 9, prove that any two straight lines drawn at right angles to two given intersecting straight lines must intersect.
3. Prove that a straight line drawn at right angles to a given straight line must intersect all straight lines which are not at right angles to the given straight line.

## PROPOSITION 18.

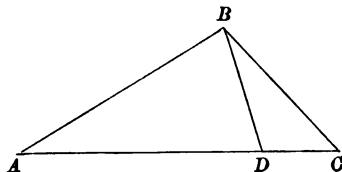
*When two sides of a triangle are unequal, the greater side has the greater angle opposite to it.*

Let  $ABC$  be a triangle, of which the side  $AC$  is greater than the side  $AB$ :

it is required to prove that the angle  $ABC$  is greater than the angle  $ACB$ .

CONSTRUCTION. From  $AC$  the greater of the two sides cut off  $AD$  equal to  $AB$  the less. (Prop. 3.)

Draw  $BD$ .



PROOF. Because  $ADB$  is an exterior angle of the triangle  $BDC$ ,

it is greater than the interior opposite angle  $DCB$ . (Prop. 16.)

And because  $AB$  is equal to  $AD$ ,  
the angle  $ADB$  is equal to the angle  $ABD$ . (Prop. 5.)

Therefore the angle  $ABD$  is greater than the angle  $ACB$ .

But the angle  $ABC$  is greater than the angle  $ABD$ ;  
therefore the angle  $ABC$  is greater than the angle  $ACB$ .

Wherefore, *when two sides &c.*

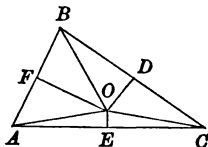
## ADDITIONAL PROPOSITION.

*The straight lines drawn at right angles to the sides of a triangle at their middle points meet in a point.*

Let  $ABC$  be a triangle and  $D, E, F$  the middle points of the sides  $BC, CA, AB$ .

Draw  $EO, FO$ \* at right angles to  $CA, AB$ .

Draw  $OA, OB, OC, OD$ .



Because in the triangles  $AEO, CEO$ ,  
 $AE$  is equal to  $CE$ ,  $EO$  common, and the angle  $AEO$  is equal to the angle  $CEO$ ,

the triangles are equal in all respects; (Prop. 4.)

therefore  $AO$  is equal to  $CO$ .

Similarly it can be proved that  $AO$  is equal to  $BO$ ;

therefore  $BO$  is equal to  $CO$ .

Next because in the triangles  $BOD, COD$ ,

$BO$  is equal to  $CO$  and  $BD$  to  $CD$  and  $OD$  is common,

the triangles are equal in all respects; (Prop. 8.)

therefore the angle  $BDO$  is equal to the angle  $CDO$ ,

and  $OD$  is at right angles to  $BC$ .

Wherefore the straight line drawn at right angles to  $BC$  at its middle point  $D$  passes through  $O$ , the intersection of the straight lines drawn at right angles to the other two sides at their middle points.

## EXERCISES.

1.  $ABC$  is a triangle and the angle  $A$  is bisected by a straight line which meets  $BC$  at  $D$ ; shew that  $BA$  is greater than  $BD$ , and  $CA$  greater than  $CD$ .

2. Prove that, if  $D$  be any point in the base  $BC$  between  $B$  and  $C$  of an isosceles triangle  $ABC$ ,  $AD$  is less than  $AB$ .

3. Prove that, if  $AB, AC, AD$  be equal straight lines, and  $AC$  fall within the angle  $BAD$ ,  $BD$  is greater than either  $BC$  or  $CD$ .

4.  $ABCD$  is a quadrilateral of which  $AD$  is the longest side and  $BC$  the shortest; shew that the angle  $ABC$  is greater than the angle  $ADC$ , and that the angle  $BCD$  is greater than the angle  $BAD$ .

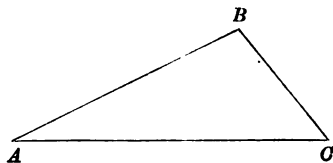
5. If the angle  $C$  of a triangle be equal to the sum of the angles  $A$  and  $B$ , the side  $AB$  is equal to twice the straight line joining  $C$  to the middle point of  $AB$ .

\* We assume that the straight lines drawn at right angles to  $CA, AB$  at  $E$  and  $F$  meet. (See Exercise 2, page 51.)

## PROPOSITION 19.

*When two angles of a triangle are unequal, the greater angle has the greater side opposite to it.*

Let  $ABC$  be a triangle, of which the angle  $ABC$  is greater than the angle  $ACB$ :  
it is required to prove that the side  $AC$  is greater than the side  $AB$ .



PROOF.  $AC$  must be either less than, equal to, or greater than  $AB$ .

If  $AC$  were less than  $AB$ ,  
the angle  $ABC$  would be less than the angle  $ACB$ ; (Prop. 18.)  
but it is not;

therefore  $AC$  is not less than  $AB$ .

If  $AC$  were equal to  $AB$ ,  
the angle  $ABC$  would be equal to the angle  $ACB$ ; (Prop. 5.)  
but it is not;

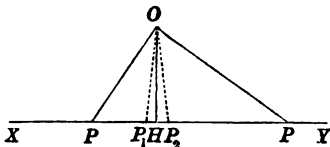
therefore  $AC$  is not equal to  $AB$ .

Therefore  $AC$  must be greater than  $AB$ .

Wherefore, *when two angles &c.*



We leave it to the student to prove that, while a point  $P$  is moving along a straight line  $XY$ , the distance  $OP$  of the point  $P$  from a fixed point  $O$  outside the line is decreasing when  $P$  is moving towards  $H$  the foot of the perpendicular from  $O$  on the line, and that  $OP$  is increasing when  $P$  is moving away from  $H$ . Assuming the



truth of this proposition, it follows that  $OH$  is *less* than each of the two straight lines  $OP_1$ ,  $OP_2$ , where  $P_1$ ,  $P_2$  are two positions of the point  $P$  close to  $H$  on either side of it. For this reason we say that  $OH$  is a **minimum** value of  $OP$ .

In the same way, if a geometrical quantity vary continuously, its magnitude in a position, where it is *greater* than in the positions close to it on either side, is called a **maximum** value.

It will be seen that, if a quantity vary continuously, there must be between any two *equal* values of the quantity at least one **maximum** or **minimum** value.

#### EXERCISES.

1. Prove that the hypotenuse of a right-angled triangle is greater than either of the other sides.
2. The base of a triangle is divided into two parts by the perpendicular from the opposite vertex: prove that each part of the base is less than the adjacent side of the triangle.
3. A straight line drawn from the vertex of an isosceles triangle to any point in the base produced is greater than either of the equal sides.
4. If  $D$  be any point in the side  $BC$  of a triangle  $ABC$ , then the greater of the sides  $AB$ ,  $AC$  is greater than  $AD$ .
5. The perpendicular is the shortest straight line which can be drawn from a given point to a given straight line; and, of any two others, that which makes the smaller angle with the perpendicular is the shorter.
6. The base of a triangle whose sides are unequal is divided into two parts by the straight line bisecting the vertical angle: prove that the greater part is adjacent to the greater side.

## PROPOSITION 20.

*The sum of any two sides of a triangle is greater than the third side.*

Let  $ABC$  be a triangle :

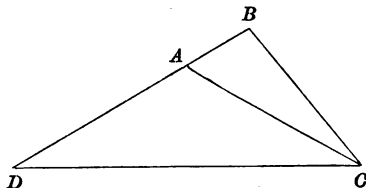
it is required to prove that the sum of any two sides of it is greater than the third side;

namely, the sum of  $CA$ ,  $AB$  greater than  $BC$  ;

the sum of  $AB$ ,  $BC$  greater than  $CA$  ;

the sum of  $BC$ ,  $CA$  greater than  $AB$ .

CONSTRUCTION. Produce any side  $BA$  to  $D$ ,  
making  $AD$  equal to  $AC$ . (Prop. 3.)  
Draw  $DC$ .



PROOF. Because  $AC$  is equal to  $AD$ ,  
the angle  $ADC$  is equal to the angle  $ACD$ . (Prop. 5.)

But the angle  $BCD$  is greater than the angle  $ACD$ .

Therefore the angle  $BCD$  is greater than the angle  $BDC$ .

And because in the triangle  $BCD$ ,

the angle  $BCD$  is greater than the angle  $BDC$  ;

$BD$  is greater than  $BC$ . (Prop. 19.)

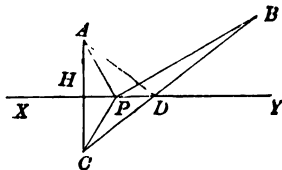
Now because  $DA$  is equal to  $AC$ ,  
 $BD$ , which is the sum of  $BA$ ,  $AD$ , is equal to the sum of  
 $CA$ ,  $AB$ .

Therefore the sum of  $CA$ ,  $AB$  is greater than  $BC$ .

Similarly it can be proved that the sum of  $AB$ ,  $BC$   
is greater than  $CA$  ; and that the sum of  $BC$ ,  $CA$  is greater  
than  $AB$ .

Wherefore, *the sum of any two sides &c.*

The result of this proposition enables us to solve a great number of problems, of which the following is a specimen—*To find in a given straight line  $XY$  a point  $P$  such that the sum of its distances  $PA, PB$  from two given points  $A, B$  is a minimum.*



If the points  $A, B$  be on opposite sides of  $XY$ , the straight line  $AB$  intersects  $XY$  in the point required.

If  $A, B$  be on the same side of  $XY$ , draw  $AH$  perpendicular to  $XY$ ; produce  $AH$  to  $C$ , so that  $HC$  is equal to  $HA$ .

Take any point  $P$  in  $XY$ .

Draw  $BDC, DA, PA, PB, PC$ . Then it is easily proved that  $AP$  is equal to  $CP$ , and  $AD$  to  $CD$ .

Therefore the sum of  $AP, PB$  is equal to the sum of  $CP, PB$ , and this is a *minimum* when  $P$  coincides with  $D$ . (Prop. 20.)

Therefore  $D$  is the point required.

From the diagram it is seen that the angle  $BDY$  is equal to the angle  $CDX$ , which is equal to the angle  $ADX$ .

It appears therefore that when the sum of  $PA, PB$  is a *minimum*, the lines  $PA, PB$  make equal angles with  $XY$ .

### EXERCISES.

1. Prove that any three sides of any quadrilateral are greater than the fourth side.

2. If  $D$  be any point within a triangle  $ABC$ , the sum of  $DA, DB, DC$  is greater than half the perimeter of the triangle.

3. The sum of the four sides of any quadrilateral is greater than the sum of its two diagonals.

4. In a convex quadrilateral the sum of the diagonals is greater than the sum of either pair of opposite sides.

5.  $D$  is the middle point of  $BC$  the base of an isosceles triangle  $ABC$ , and  $E$  any point in  $AC$ . Prove that the difference of  $BD, DE$  is less than the difference of  $AB, AE$ .

6. The two sides of a triangle are together greater than twice the straight line drawn from the vertex to the middle point of the base.

7. Find in a given straight line a point such that the difference of its distances from two fixed points is a *maximum*.

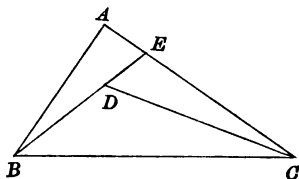
## PROPOSITION 21.

*If from the ends of the side of a triangle there be drawn two straight lines to a point within the triangle, the sum of these lines is less than the sum of the other two sides of the triangle, but they contain a greater angle.*

Let  $ABC$  be a triangle; and from  $B, C$ , the ends of the side  $BC$ , let the two straight lines  $BD, CD$  be drawn to a point  $D$  within the triangle:

it is required to prove that the sum of  $BD, DC$  is less than the sum of  $BA, AC$ , but the angle  $BDC$  is greater than the angle  $BAC$ .

CONSTRUCTION. Produce  $BD$  to meet  $AC$  at  $E$ .



PROOF. The sum of the two sides  $BA, AE$  of the triangle  $BAE$  is greater than the third side  $BE$ . (Prop. 20.)

To each of these unequals add  $EC$ ;

then the sum of  $BA, AC$  is greater than the sum of  $BE, EC$ .

Again, the sum of the two sides  $CE, ED$  of the triangle  $CED$  is greater than the third side  $CD$ .

To each of these unequals add  $DB$ ;

then the sum of  $CE, EB$  is greater than the sum of  $CD, DB$ .

And it has been proved that the sum of  $BA, AC$  is greater than the sum of  $BE, EC$ ;

therefore the sum of  $BA, AC$  is greater than the sum of  $BD, DC$ .

Again, the exterior angle  $BDC$  of the triangle  $CDE$  is greater than the interior opposite angle  $CED$ . (Prop. 16.)

And the exterior angle  $CEB$  of the triangle  $ABE$  is greater than the interior opposite angle  $BAE$ ;

therefore the angle  $BDC$  is greater than the angle  $BAC$ .

Wherefore, *if from the ends &c.*

## EXERCISES.

1. If  $D$  be any point within a triangle  $ABC$ , the sum of  $DA$ ,  $DB$ ,  $DC$  is less than the perimeter of the triangle and greater than half the perimeter.

2. Prove that the perimeter of a triangle is less than the perimeter of any triangle which is drawn completely surrounding it.

3. If two triangles have a common base and equal vertical angles, the vertex of each triangle lies outside the other triangle.

4. If from the angles of a triangle  $ABC$ , straight lines  $AOD$ ,  $BOE$ ,  $COF$  be drawn through a point  $O$  within the triangle to meet the opposite sides, the perimeter of the triangle  $ABC$  is greater than two-thirds of the sum of  $AD$ ,  $BE$ ,  $CF$ .

5.  $ABD$ ,  $ACD$  are two triangles on the same side of  $AD$  in which  $AC$  is greater than  $AB$ . Prove that, if the angles  $ABD$ ,  $ACD$  be both right angles or be equal obtuse angles, then  $BD$  is greater than  $DC$ .

## PROPOSITION 22.

*To construct a triangle having its sides equal to three given straight lines.*

Let  $AB$ ,  $CD$ ,  $EF$  be the three given lines :  
it is required to construct a triangle whose sides are equal  
to  $AB$ ,  $CD$ ,  $EF$ .

CONSTRUCTION. Produce one of the given lines  $CD$  both  
ways,

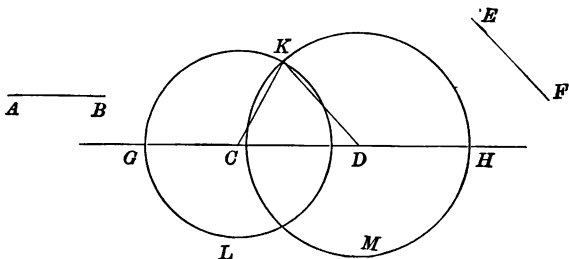
and cut off  $CG$  equal to  $AB$ , (Prop. 3.)  
and  $DH$  to  $EF$ .

With  $C$  as centre and  $CG$  as radius describe the circle  $GKL$ ,  
and with  $D$  as centre and  $DH$  as radius describe the circle  
 $HKM$ .

Let these circles intersect in  $K$  :

Draw  $CK$ ,  $DK$  :

then  $CKD$  is a triangle drawn as required.



PROOF. Because  $C$  is the centre of the circle  $GKL$ ,  
 $CK$  is equal to  $CG$  ;  
and  $CG$  is equal to  $AB$ . (Constr.)

Therefore  $CK$  is equal to  $AB$ .

Again, because  $D$  is the centre of the circle  $HKM$ ,

$DK$  is equal to  $DH$  ;

and  $DH$  is equal to  $EF$ . (Constr.)

Therefore  $DK$  is equal to  $EF$ .

Therefore the three lines  $KC$ ,  $CD$ ,  $DK$  are equal to the  
three  $AB$ ,  $CD$ ,  $EF$  respectively.

Wherefore, the triangle  $KCD$  has been constructed having  
its sides equal to the three given straight lines  $AB$ ,  $CD$ ,  $EF$ .

It may be observed that it is not possible to construct a triangle which shall have its sides equal to *any* three given straight lines. In Proposition 20 it has been proved that any two sides of a triangle are together greater than the third side. It follows therefore that it is impossible to construct a triangle having its sides equal to three given straight lines, except when the given straight lines are such that *any two of them are greater than the third or the greatest line is less than the sum of the other two.*

We see therefore that in this proposition we have to solve a problem, which admits of solution only when the *given* lines satisfy a certain condition.

We shall meet with many other problems in which the geometrical quantities *given* in the problem (for that reason generally called the **data**), must satisfy some condition in order that the problem may admit of solution. It will be a useful exercise for the student to investigate such conditions when they exist.

#### EXERCISES.

1. Prove that the two circles drawn in the construction of Proposition 22 will always intersect, provided that the sum of any two of the given straight lines is greater than the third.
2. How many different shaped triangles could be made of 8 different lines whose lengths are respectively 2, 2, 2, 3, 3, 4, 4, 5 inches?
3. Construct a right-angled triangle, having given the hypotenuse and one side.
4. Construct a quadrilateral equal in all respects to a given quadrilateral.

## PROPOSITION 23.

*From a given point in a given straight line to draw a straight line making with the given straight line an angle equal to a given angle.*

Let  $ABC$  be the given straight line,  $B$  the given point in it, and  $DEF$  the given angle:

it is required to draw from  $B$  a straight line making with  $ABC$  an angle equal to the angle  $DEF$ .

CONSTRUCTION. In  $ED$ ,  $EF$  take any points  $G$ ,  $H$ , and draw  $GH$ .

From  $BC$  cut off  $BK$  equal to  $EH$ , (Prop. 3.)

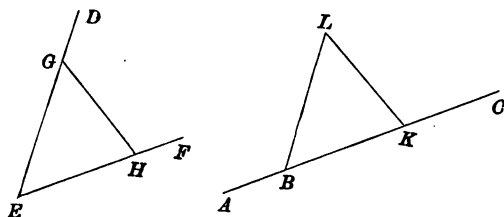
and construct the triangle  $LBK$ ,

having the side  $BK$  equal to  $EH$ ,

$BL$  equal to  $EG$ ,

and  $KL$  equal to  $GH$ : (Prop. 22.)

then  $BL$  is a straight line drawn as required.



PROOF. Because in the triangles  $LBK$ ,  $EGH$ ,  
 $KB$  is equal to  $HE$ ,  
 $BL$  to  $EG$ ,  
and  $LK$  to  $GH$ ,

the triangles are equal in all respects; (Prop. 8.)  
therefore the angle  $KBL$  (or  $CBL$ ) is equal to the angle  
 $HEG$  (or  $FED$ ).

Wherefore, from the given point  $B$  in the given straight line  $ABC$  a straight line  $BL$  has been drawn making with the straight line  $ABC$  an angle  $KBL$  equal to the given angle  $FED$ .



**EXERCISES.**

1. Construct a triangle, having given the base and each of the angles at the base.
2. Make an angle double of a given angle.
3. If one angle of a triangle be equal to the sum of the other two, the triangle can be divided into two isosceles triangles.
4. Construct a triangle, having given the base, one of the angles at the base, and the sum of the sides.

## PROPOSITION 24.

*If two sides of one triangle be equal to two sides of another and the angle contained by the two sides of the one be greater than the angle contained by the two sides of the other, the third side of the one is greater than the third side of the other.*

Let  $ABC$ ,  $DEF$  be two triangles, in which  $AB$  is equal to  $DE$  and  $AC$  to  $DF$ , and the angle  $BAC$  is greater than the angle  $EDF$ :

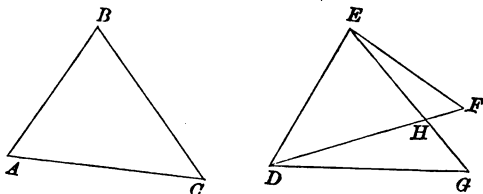
it is required to prove that the third side  $BC$  is greater than the third side  $EF$ .

CONSTRUCTION. Of the two sides  $DE$ ,  $DF$  let  $DF$  be one which is not less than the other. From the point  $D$  in the straight line  $DE$ , draw  $DG$  making with  $DE$

the angle  $EDG$  equal to the angle  $BAC$ , (Prop. 23.)

and make  $DG$  equal to  $DF$ . (Prop. 3.)

Draw  $EG$  meeting  $DF$  in  $H$ .



PROOF. Because  $DF$  is not less than  $DE$ ,  
and  $DG$  is equal to  $DF$ ,  
 $DG$  is not less than  $DE$ .

And because in the triangle  $DEG$ ,  
 $DG$  is not less than  $DE$ ,

the angle  $DEG$  is not less than the angle  $DGE$ .

(Props. 5 and 18.)

Next, because  $DHG$  is the exterior angle of the triangle  $DEH$ ,  
it is greater than the interior opposite angle  $DEG$ .

(Prop. 16.)

Therefore the angle  $DHG$  is greater than the angle  $DGH$ .

And because in the triangle  $DHG$ ,

the angle  $DHG$  is greater than the angle  $DGH$ ,

$DG$  is greater than  $DH$ . (Prop. 19.)

But  $DG$  is equal to  $DF$ .

Therefore  $DF$  is greater than  $DI$ ,  
or the point  $F$  lies outside the triangle  $DEI$ .

Next because the sum of  $DH$ ,  $HG$ , two sides of the triangle  $DHG$ , is greater than the third side  $DI$ ,  
and the sum of  $FH$ ,  $HE$ , two sides of the triangle  $HFF$ , is greater than the third side  $EF$ ;  
the sum of  $DH$ ,  $HG$ ,  $FH$ ,  $HE$  is greater than the sum of  $DI$ ,  $EF$ ;

i.e. the sum of  $DF$ ,  $EG$  is greater than the sum of  $DI$ ,  $EF$ .

Take away the equals  $DF$ ,  $DG$ ;

then  $EG$  is greater than  $EF$ .

Now the triangles  $EDG$ ,  $BAC$  are equal in all respects,  
(Prop. 4.)

Therefore  $BC$ , which is equal to  $EG$ , is greater than  $EF$ .

Wherefore, if two sides &c.

### EXERCISES.

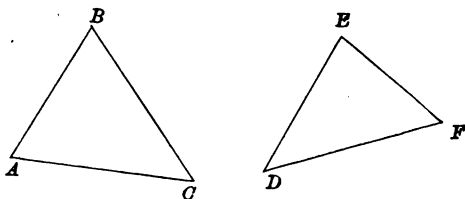
A point  $P$  moves along the circumference of a circle from one extremity  $A$  of a diameter  $AB$  to the other extremity  $B$ ; prove that throughout the motion

- (a)  $AP$  is increasing and  $BP$  is decreasing;
- (b) if  $O$  be any point in  $AB$  nearer  $A$  than  $B$ ,  $OP$  is increasing;
- (c) if  $O$  be any point in  $BA$  produced,  $OP$  is increasing.

## PROPOSITION 25.

*If two sides of one triangle be equal to two sides of another, and the third side of the one be greater than the third side of the other, the angle opposite to the third side of the one is greater than the angle opposite to the third side of the other.*

Let  $ABC$ ,  $DEF$  be two triangles, in which  $AB$  is equal to  $DE$ , and  $AC$  to  $DF$ , and  $BC$  is greater than  $EF$ : it is required to prove that the angle  $BAC$  is greater than the angle  $EDF$ .



PROOF. The angle  $BAC$  must be either greater than, equal to, or less than the angle  $EDF$ .

If the angle  $BAC$  were equal to the angle  $EDF$ ,  
 $BC$  would be equal to  $EF$ ; (Prop. 4.)  
 but it is not;

therefore the angle  $BAC$  is not equal to the angle  $EDF$ .

Again, if the angle  $BAC$  were less than the angle  $EDF$ ,  
 $BC$  would be less than  $EF$ ; (Prop. 24.)  
 but it is not;

therefore the angle  $BAC$  is not less than the angle  $EDF$ .

Therefore the angle  $BAC$  is greater than the angle  $EDF$ .

Wherefore, *if two sides &c.*

## EXERCISES.

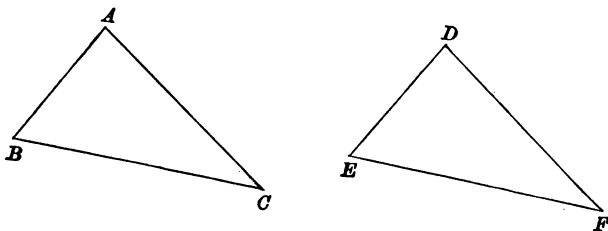
1. If  $D$  be the middle point of the side  $BC$  of a triangle  $ABC$ , in which  $AC$  is greater than  $AB$ , the angle  $ADC$  is an obtuse angle.
2. If in the sides  $AB, AC$  of a triangle  $ABC$ , in which  $AC$  is greater than  $AB$ , points  $D, E$  be taken such that  $BD, CE$  are equal,  $CD$  is greater than  $BE$ .
3. If in the sides  $AB, AC$  produced of a triangle  $ABC$ , in which  $AC$  is greater than  $AB$ , points  $D, E$  be taken such that  $BD, CE$  are equal,  $BE$  is greater than  $CD$ .
4. If in the side  $AB$  and the side  $AC$  produced of a triangle  $ABC$  points  $D$  and  $E$  be taken, such that  $BD, CE$  are equal,  $BE$  is greater than  $CD$ .

## PROPOSITION 26. PART 1.

*If two triangles have two angles of the one equal to two angles of the other, and the side adjacent to the angles in the one equal to the side adjacent to the angles in the other, the triangles are equal in all respects.*

Let  $ABC$ ,  $DEF$  be two triangles, in which the angle  $ABC$  is equal to the angle  $DEF$ , and the angle  $BCA$  is equal to the angle  $EFD$ , and the side  $BC$  adjacent to the angles  $ABC$ ,  $BCA$  is equal to the side  $EF$  adjacent to the angles  $DEF$ ,  $EFD$ :

it is required to prove that the triangles  $ABC$ ,  $DEF$  are equal in all respects.



PROOF. Because the sides  $BC$ ,  $EF$  are equal, it is possible to shift the triangle  $ABC$ , so that  $BC$  coincides with  $EF$ ,  $B$  with  $E$  and  $C$  with  $F$ ,  
(Test of Equality, page 5.)

and the triangles are on the same side of  $EF$ .

If this be done,

because  $BC$  coincides with  $EF$ ,

and the angle  $ABC$  is equal to the angle  $DEF$ ,

$BA$  must coincide in direction with  $ED$ .

Similarly it may be proved that  $CA$  must coincide in direction with  $FD$ .

Therefore the point  $A$ , which is the intersection of  $BA$ ,  $CA$ , must coincide with  $D$ , which is the intersection of  $ED$ ,  $FD$ .

Next, because  $A$  coincides with  $D$ , and  $B$  with  $E$ ,

$AB$  must coincide with  $DE$ . (Post. 2.)

Similarly  $AC$  must coincide with  $DF$ .

Therefore the triangle  $ABC$  coincides with the triangle  $DEF$ , and is equal to it in all respects.

Wherefore, *if two triangles &c.*

## EXERCISES.

1. If  $AD$  be the bisector of the angle  $BAC$ , and  $BDC$  be drawn at right angles to  $AD$ ,  $AB$  is equal to  $AC$ .

2.  $AB, AC$  are any two straight lines meeting at  $A$ : through any point  $P$  draw a straight line meeting them at  $E$  and  $F$ , such that  $AE$  may be equal to  $AF$ .

3. If upon the same base  $AB$  two triangles  $BAC, ABD$  be constructed, having the angle  $BAC$  equal to  $ABD$ , and  $ABC$  equal to  $BAD$ , then the triangles  $BDC, ACD$  are equal in all respects.

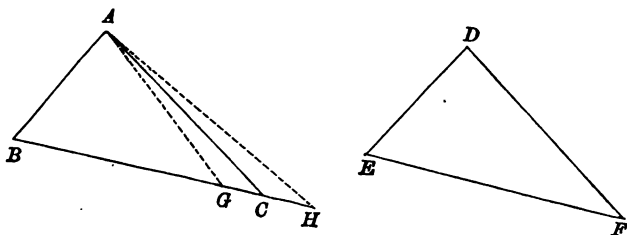
4. If the opposite sides of a quadrilateral be equal, the diagonals bisect each other.

5. If the straight line bisecting the vertical angle of a triangle be at right angles to the base, the triangle is isosceles.

## PROPOSITION 26. PART 2.

*If two triangles have two angles of the one equal to two angles of the other, and the sides opposite to a pair of equal angles equal, the triangles are equal in all respects.*

Let  $ABC$ ,  $DEF$  be two triangles, in which the angle  $ABC$  is equal to the angle  $DEF$ , and the angle  $BCA$  equal to the angle  $EFD$ , and  $BA$  the side opposite to the angle  $BCA$  is equal to  $ED$  the side opposite to the angle  $EFD$ : it is required to prove that the triangles  $ABC$ ,  $DEF$  are equal in all respects.



PROOF. Because the sides  $AB$ ,  $DE$  are equal, it is possible to shift the triangle  $DEF$ , so that  $DE$  coincides with  $AB$ ,  $D$  with  $A$  and  $E$  with  $B$ , and the triangles are on the same side of  $AB$ .

If this be done,

because  $ED$  coincides with  $BA$ ,  
and the angle  $DEF$  is equal to the angle  $ABC$ ,  
 $EF$  must coincide in direction with  $BC$ .

Now  $F$  cannot coincide with any point  $G$  in  $BC$ , since the angle  $AGB$  the exterior angle of the triangle  $AGC$  is greater than the interior and opposite angle  $ACB$ ,  
(Prop. 16.)

which is equal to the angle  $DFE$ .

Again  $F$  cannot coincide with any point  $H$  in  $BC$  produced, since the interior and opposite angle  $AHB$  of the triangle  $ACH$  is less than the exterior angle  $ACB$ , (Prop. 16.)  
which is equal to the angle  $DFE$ .



Therefore  $F$  must coincide with  $C$ ,  
 $EF$  with  $BC$ , and  $DF$  with  $AC$ ;  
 therefore the triangle  $DEF$  coincides with the triangle  $ABC$ ,  
 and is equal to it in all respects.

Wherefore, if two triangles &c.

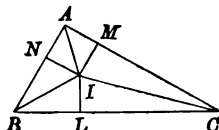
### ADDITIONAL PROPOSITION.

*The straight lines, which bisect the angles of a triangle, meet in a point.*

Let  $ABC$  be a triangle.

Bisect the angles  $ABC$ ,  $BCA$  by the straight lines  $BI$ ,  $CI$ \*.

Draw  $IL$ ,  $IM$ ,  $IN$  perpendicular to the sides.



Because in the triangles  $IBN$ ,  $IBL$   
 the angle  $IBN$  is equal to the angle  $IBL$ ,  
 and the angle  $INB$  to the angle  $ILB$ ,  
 and  $BI$  is common,

the triangles are equal in all respects: (Prop. 26, Part 2.)  
 therefore  $IN$  is equal to  $IL$ .

Similarly it can be proved that  $IM$  is equal to  $IL$ :  
 therefore  $IN$  is equal to  $IM$ .

Next because in the right-angled triangles  $IAN$ ,  $IAM$   
 the hypotenuse  $IA$  is common,  
 and  $IN$  is equal to  $IM$ ,

the triangles are equal in all respects: (Exercise 5, page 49.)  
 therefore the angle  $IAN$  is equal to the angle  $IAM$ ,  
 and  $IA$  is the bisector of the angle  $BAC$ .

Therefore the bisector of the angle  $BAC$  passes through the intersection  
 of the bisectors of the angles  $ABC$ ,  $BCA$ .

### EXERCISES.

1. The perpendiculars let fall on two sides of a triangle from any point in the straight line bisecting the angle between them are equal to each other.

2. In a given straight line find a point such that the perpendiculars drawn from it to two given straight lines which intersect are equal.

3. Through a given point draw a straight line such that the perpendiculars on it from two given points may be on opposite sides of it and equal to each other.

\* It is assumed that these lines intersect.

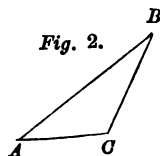
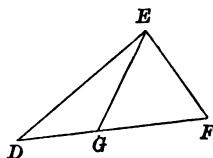
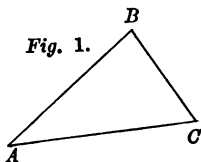
## PROPOSITION 26 A.

*If two triangles have two sides equal to two sides, and the angles opposite to one pair of equal sides equal, the angles opposite to the other pair are either equal or supplementary.*

Let  $ABC$ ,  $DEF$  be two triangles, in which  $AB$  is equal to  $DE$ , and  $BC$  to  $EF$ , and the angle  $BAC$  is equal to the angle  $EDF$ :

it is required to prove that the angles  $ACB$ ,  $DFE$  are either equal or supplementary. (Def. page 45.)

Of the two sides  $AC$ ,  $DF$ , let  $AC$  be not greater than  $DF$ .



**PROOF.** Because the sides  $AB$ ,  $DE$  are equal, it is possible to shift the triangle  $ABC$ ,

so that  $AB$  coincides with  $DE$ ,

$A$  with  $D$  and  $B$  with  $E$ , (Test of Equality, page 5.)

and so that the two triangles  $ABC$ ,  $DEF$  are on the same side of  $DE$ .

If this be done,

because  $AB$  coincides with  $DE$ ,

and the angle  $BAC$  is equal to the angle  $EDF$ ,

$AC$  must coincide in direction with  $DF$ .

Because  $AC$  is not greater than  $DF$ ,

$C$  must coincide either (1) with  $F$  or (2) with  $G$  some point in  $DF$ .

(Fig. 1.) If  $C$  coincide with  $F$ ,

then  $BC$  coincides with  $EF$ , (Post. 2.)

and the triangle  $ABC$  with the triangle  $DEF$ ,

and the two triangles are equal in all respects;

therefore the angle  $ACB$  is equal to the angle  $DFE$ .

(Fig. 2.) Again, if  $C$  coincide with  $G$ ,

because  $BC$  is equal to  $EG$ , and  $EF$  is equal to  $BC$ ,

$EG$  is equal to  $EF$ .

And because in the triangle  $EFG$ ,

$EG$  is equal to  $EF$ ,

the angle  $EFG$  is equal to the angle  $EGF$ . (Prop. 5.)

Now the angles  $DGE$ ,  $EGF$  are together equal to two right angles, i.e. are supplementary; (Prop. 13.)

therefore the angles  $DGE$ ,  $EFG$  are supplementary;

and the angle  $DGE$  is equal to the angle  $ACB$ ;

therefore the angles  $ACB$ ,  $DFE$  are supplementary.

Wherefore, if two triangles &c.

**COROLLARY.** *When two triangles have two sides equal to two sides, and the angles opposite to one pair of equal sides equal to one another, they are equal in all respects, provided that of the angles opposite to the second pair of equal sides,*

- (1) *each be less than a right angle,*
- (2) *each be greater than a right angle,*
- or (3) *one of them be a right angle.*

### EXERCISES.

1. If the straight line bisecting the vertical angle of a triangle also bisect the base, the triangle is isosceles.

2. If two given straight lines intersect, and a point be taken equally distant from each of them, it lies on one or other of the two straight lines which bisect the angles between the given straight lines.

3. Prove that two right-angled triangles are equal in all respects, if the hypotenuse and a side of the one be respectively equal to the hypotenuse and a side of the other.

4. If two exterior angles of a triangle be bisected, and from the point of intersection of the bisecting lines a straight line be drawn to the third angle, it bisects that angle.

5. If two triangles have two sides equal to two sides, and the angles opposite to the greater sides equal, the triangles are equal in all respects.

6. Construct a triangle having given two sides and the angle opposite to one of them. Is this always possible?

## ON EQUAL TRIANGLES.

It is in many cases convenient to denominate the sides  $BC$ ,  $CA$ ,  $AB$  of a triangle  $ABC$  by the small letters  $a$ ,  $b$ ,  $c$  respectively. Here  $a$ ,  $b$ ,  $c$  stand for the sides of the triangle opposite to the angles  $A$ ,  $B$ ,  $C$  respectively.

Using this notation we may sum up the results of Propositions 4, 8, 26 Part 1, 26 Part 2, and 26 A as follows:

Two triangles  $ABC$ ,  $A'B'C'$  are equal in all respects,

(I) if  $a = a'$ ,  $b = b'$ , and  $C = C'$ , (Prop. 4.)

(II) if  $a = a'$ ,  $b = b'$ , and  $c = c'$ , (Prop. 8.)

(III) if  $A = A'$ ,  $B = B'$ , and  $c = c'$ , (Prop. 26; Part 1.)

(IV) if  $A = A'$ ,  $B = B'$ , and  $a = a'$ , (Prop. 26, Part 2.)

(V) if  $a = a'$ ,  $b = b'$ , and  $A = A'$ , and if in addition

(1)  $B$  and  $B'$  be each less than a right angle,

or (2)  $B$  and  $B'$  be each greater than a right angle,

or (3) either  $B$  or  $B'$  be a right angle. (Prop. 26 A.)

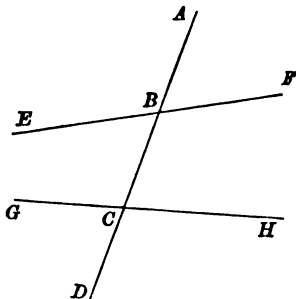
The six quantities, the angles  $A$ ,  $B$ ,  $C$ , and the sides  $a$ ,  $b$ ,  $c$ , are often denominated the **parts** of the triangle  $ABC$ .

It will be observed that the equality of three pairs of *parts* is always required to ensure the equality in all respects of two triangles, but that the equality of three pairs of *parts* is not always sufficient.

By the theorem of Proposition 32 it can be shewn that, if any two of the equations  $A = A'$ ,  $B = B'$ ,  $C = C'$ , be true, the third is also true: from this we conclude that the set of equations  $A = A'$ ,  $B = B'$ ,  $C = C'$ , is insufficient to determine the equality of the triangles, and that the two cases III. and IV. are virtually the same.

ON THE ANGLES MADE BY ONE STRAIGHT LINE WITH TWO OTHERS.

When a straight line  $ABCD$  intersects two other straight lines  $EBF$ ,  $GCH$ ,



the angles  $ABE$ ,  $ABF$ ,  $DCG$ ,  $DCH$  *outside* the two lines  $EF$ ,  $GH$  are called **exterior** angles ;  
the angles  $CBE$ ,  $CBF$ ,  $BCG$ ,  $BCH$  *inside* the two lines  $EF$ ,  $GH$  are called **interior** angles ;  
a pair of interior angles on opposite sides of  $ABCD$  are called **alternate** angles.

There are two pairs of *alternate* angles in the diagram,  $EBC$ ,  $BCH$  ;  
 $CBF$ ,  $BCG$ .

A pair of angles, one at  $B$  and the other at  $C$ , one exterior and the other interior, on the same side of  $ABCD$  are sometimes called *corresponding* angles.

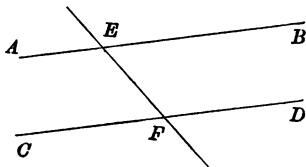
There are four pairs of corresponding angles,  $ABF$ ,  $BCH$  ;  $ABE$ ,  $BCG$  ;  
 $DCH$ ,  $CBF$  ; and  $DCG$ ,  $CBE$ , the first angle in each pair being an exterior angle, and the second the interior.

## PROPOSITION 27.

*If a straight line, meeting two other straight lines in the same plane, make two alternate angles equal, the two straight lines are parallel.*

Let the straight line  $EF$ , meeting the two straight lines  $AB$ ,  $CD$  in the same plane, make the alternate angles  $AEF$ ,  $EFD$  equal to one another :

it is required to prove that  $AB$ ,  $CD$  are parallel.



PROOF.  $AB$ ,  $CD$  cannot meet when produced beyond  $B$  and  $D$  ; for if they did, the exterior angle  $AEF$  of the triangle formed by them and  $EF$  would be greater than the interior opposite angle  $EFD$  (Prop. 16.);  
but it is not.

Similarly it can be proved that  $AB$ ,  $CD$  cannot meet when produced beyond  $A$  and  $C$ .

But those straight lines in the same plane which do not meet however far they may be produced both ways, are parallel. (Def. 9.)

Therefore  $AB$ ,  $CD$  are parallel.

Wherefore, if a straight line &c.

**EXERCISES.**

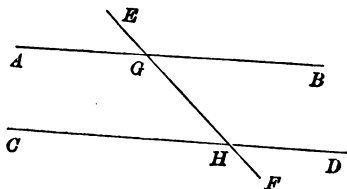
1. No two straight lines drawn from two angles of a triangle and terminated by the opposite sides can bisect one another.
2. Two straight lines at right angles to the same straight line are parallel.
3. Prove Proposition 27 by the method of superposition.

## PROPOSITION 28.

*If a straight line intersecting two other straight lines, make an exterior angle equal to the interior and opposite angle on the same side of the line; or if it make two interior angles on the same side together equal to two right angles, the two straight lines are parallel.*

Let the straight line  $EF$ , intersecting the two straight lines  $AB$ ,  $CD$ ,

- (1) make the exterior angle  $EGB$  equal to the interior and opposite angle on the same side  $GHD$ ,  
 or (2) make the interior angles on the same side  $BGH$ ,  $GHD$  together equal to two right angles:  
 it is required to prove that  $AB$ ,  $CD$  are parallel.



PROOF. (1) Because the angle  $EGB$  is equal to the angle  $GHD$ ,  
 and the angle  $EGB$  is equal to the angle  $AGH$ , (Prop. 15.)  
 the angle  $AGH$  is equal to the angle  $GHD$ ;  
 and they are alternate angles;  
 therefore  $AB$ ,  $CD$  are parallel. (Prop. 27.)

(2) Because the angles  $BGH$ ,  $GHD$  are together equal to two right angles,  
 and the angles  $AGH$ ,  $BGH$  are together equal to two right angles,  
 angles, (Prop. 13.)  
 the angles  $AGH$ ,  $BGH$  are together equal to the angles  $BGH$ ,  $GHD$ .

Take away the common angle  $BGH$ ;  
 then the angle  $AGH$  is equal to the angle  $GHD$ ;  
 and they are alternate angles;  
 therefore  $AB$ ,  $CD$  are parallel. (Prop. 27.)

Wherefore, if a straight line &c.



## EXERCISES.

1. If a straight line intersecting two other straight lines make two external angles on the same side of the line together equal to two right angles, the two straight lines are parallel.

2. If a straight line intersecting two other straight lines make two external angles on opposite sides of the line equal, the two straight lines are parallel.

3. If a straight line intersecting two other straight lines make two corresponding angles equal, the two straight lines are parallel.

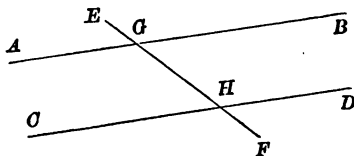
## PROPOSITION 29.

*If a straight line intersect two parallel straight lines, it makes alternate angles equal, it makes each exterior angle equal to the interior and opposite angle on the same side of the line, and it also makes interior angles on the same side together equal to two right angles.*

Let the straight line  $EF$  intersect the two parallel straight lines  $AB$ ,  $CD$ :

it is required to prove that

- (1) the alternate angles  $AGH$ ,  $GHD$  are equal,
- (2) the exterior angle  $EGB$  is equal to the interior and opposite angle  $GHD$  on the same side of  $EF$ ,
- and (3) the two interior angles  $BGH$ ,  $GHD$  on the same side of  $EF$  are together equal to two right angles.



PROOF. (1) Because  $AGH$ ,  $BGH$  are the angles which  $EF$  makes with  $AB$  on one side of it, the sum of the angles  $AGH$ ,  $BGH$  is equal to two right angles. (Prop. 13.)

Therefore, if the angles  $AGH$ ,  $GHD$  were unequal, the sum of the angles  $BGH$ ,  $GHD$  would not be equal to two right angles;

and since these are the interior angles which the straight lines  $AB$ ,  $CD$  make with  $EF$  on one side of it,

$AB$ ,  $CD$  would not be parallel. (Post. 9, page 51.)

But  $AB$ ,  $CD$  are parallel;

therefore the angle  $AGH$  is equal to the angle  $GHD$ .

(2) But the angle  $AGH$  is equal to the angle  $EGB$ ;

(Prop. 15.)

therefore the angle  $EGB$  is equal to the angle  $GHD$ .

(3) Add to each of the equal angles  $EGB$ ,  $GHD$  the angle  $BGH$ ;

then the angles  $EGB$ ,  $BGH$  are together equal to the angles  $BGH$ ,  $GHD$ .

But the angles  $EGB$ ,  $BGH$  are together equal to two right angles.

Therefore the angles  $BGH$ ,  $GHD$  are together equal to two right angles.

Wherefore, *if a straight line &c.*

COROLLARY. *All the angles of a rectangle are right angles.*  
(See Def. 19.)

#### EXERCISES.

1. Any straight line parallel to the base of an isosceles triangle makes equal angles with the sides.

2. If through any point equidistant from two parallel straight lines, two straight lines be drawn cutting the parallel straight lines, they will intercept equal portions of these parallel straight lines.

3. If the straight line bisecting an exterior angle of a triangle be parallel to a side, the triangle is isosceles.

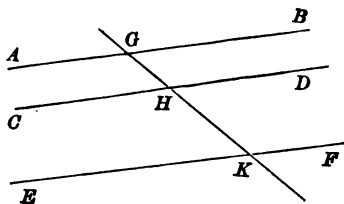
4. If  $DE$ ,  $DF$  drawn from  $D$  any point in the base  $BC$  of an isosceles triangle  $ABC$ , to meet  $AB$ ,  $AC$  in  $E$ ,  $F$  be parallel to  $AC$ ,  $AB$ , the perimeter of the parallelogram  $AEDF$  is constant.

## PROPOSITION 30.

*Straight lines parallel to the same straight line are parallel to each other.*

Let each of the straight lines  $AB$ ,  $CD$  be parallel to  $EF$ :  
it is required to prove that  $AB$ ,  $CD$  are parallel to one another.

CONSTRUCTION. Draw a straight line  $GHK$  intersecting  $AB$ ,  $CD$ ,  $EF$  in  $G$ ,  $H$ ,  $K$  respectively.



PROOF. Because  $GHK$  intersects the parallels  $AB$ ,  $EF$ ,  
the angle  $GKF$  is equal to the angle  $AGH$ . (Prop. 29.)  
Again, because  $GK$  intersects the parallels  $CD$ ,  $EF$ ,  
the angle  $GHD$  is equal to the angle  $GKF$ . (Prop. 29.)  
Therefore the angle  $AGH$  is equal to the angle  $GHD$ ;  
and they are alternate angles;  
therefore  $AB$  is parallel to  $CD$ . (Prop. 27.)

Wherefore, *straight lines &c.*

**EXERCISES.**

1. Two intersecting straight lines cannot both be parallel to the same straight line.
2. Only one straight line can be drawn through a given point parallel to a given straight line.
3. If two straight lines, each of which is parallel to a third straight line, meet, the two lines are coincident throughout their length.
4. If a straight line intersect one of two parallel straight lines, it must intersect the other.

## PROPOSITION 31.

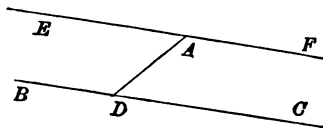
*To draw through a given point a straight line parallel to a given straight line.*

Let  $A$  be the given point, and  $BC$  the given straight line:

it is required to draw through  $A$  a straight line parallel to  $BC$ .

CONSTRUCTION. In  $BC$  take any point  $D$ , and draw  $AD$ ; from the point  $A$  in the straight line  $AD$  on the side of  $AD$  remote from  $C$  draw  $AE$  making the angle  $DAE$  equal to the angle  $ADC$ ; (Prop. 23.)

and produce the straight line  $EA$  to  $F$ :  
then  $EF$  is the straight line required.



PROOF. Because the straight line  $AD$  meets the two straight lines  $BC$ ,  $EF$ ,

and makes the alternate angles  $EAD$ ,  $ADC$  equal,

$EF$  is parallel to  $BC$ . (Prop. 27.)

Wherefore, the straight line  $EAF$  has been drawn through the given point  $A$ , parallel to the given straight line  $BC$ .

## EXERCISES.

1. Find a point  $B$  in a given straight line  $CD$ , such that, if  $AB$  be drawn to  $B$  from a given point  $A$ , the angle  $ABC$  will be equal to a given angle.
2. Draw through a given point between two intersecting straight lines a straight line so that it is bisected at the point.
3.  $ABCD$  is a quadrilateral having  $BC$  parallel to  $AD$ ; shew that its area is the same as that of the parallelogram which can be formed by drawing through the middle point of  $DC$  a straight line parallel to  $AB$ .
4.  $AC, BC$  are two given straight lines: it is required to draw a straight line from a given point  $P$  to  $AC$ , so that it is bisected by  $BC$ .
5. Construct a triangle having given two angles, and the length of the perpendicular from the third angle on the opposite side.
6. Construct a right-angled triangle, having given one side and the angle opposite.

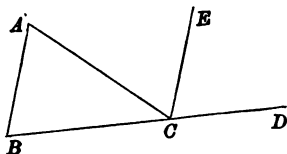
## PROPOSITION 32.

*An exterior angle of a triangle is equal to the sum of the two interior opposite angles; and the sum of the three interior angles of a triangle is equal to two right angles.*

Let  $ABC$  be a triangle:

it is required to prove that (1) the exterior angle  $ACD$  made by producing the side  $BC$  is equal to the sum of the two interior opposite angles  $CAB, ABC$ , and (2) the sum of the three interior angles  $ABC, BCA, CAB$  is equal to two right angles.

CONSTRUCTION. Through the point  $C$  draw  $CE$  parallel to  $BA$ . (Prop. 31.)



PROOF. (1) Because  $AC$  meets the parallels  $BA, CE$ , the alternate angles  $BAC, ACE$  are equal. (Prop. 29.)

Again, because  $BD$  meets the parallels  $BA, CE$ , the exterior angle  $ECD$  is equal to the interior opposite angle  $ABC$ . (Prop. 29.)

And the angle  $ACE$  was proved to be equal to the angle  $BAC$ ;

therefore the whole angle  $ACD$  is equal to the sum of the two angles  $CAB, ABC$ .

(2) To each of these equals add the angle  $BCA$ ; then the sum of the angles  $ACD, ACB$  is equal to the sum of the three angles  $ABC, BCA, CAB$ .

But the sum of the angles  $ACD, ACB$  is equal to two right angles; (Prop. 13.)

therefore also the sum of the angles  $ABC, BCA, CAB$  is equal to two right angles.

Wherefore, *An exterior angle &c.*

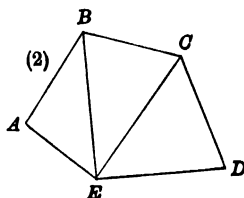
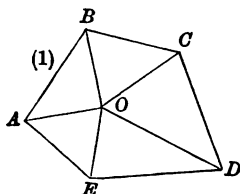


**COROLLARY.** *The sum of the interior angles of any convex rectilinear figure of  $n$  sides is less by four right angles than  $2n$  right angles.*

This may be proved in either of the following ways :

In fig. (1), where straight lines are drawn from any point  $O$  within the figure to the vertices, the angles of the  $n$  triangles so formed are equal to the angles of the figure together with the angles at  $O$ , which are equal to four right angles.

In fig. (2), where all the diagonals from one vertex  $E$  are drawn, the angles of the  $n-2$  triangles so formed are together equal to the angles of the figure.



### EXERCISES.

1. Straight lines  $AD$ ,  $BE$ ,  $CF$  are drawn within the triangle  $ABC$  making the angles  $DAB$ ,  $EBC$ ,  $FCA$  all equal to one another. If  $AD$ ,  $BE$ ,  $CF$  do not meet in a point, the angles of the triangle formed by them are equal to those of the triangle  $ABC$ .

2. Trisect a right angle.

3. Trisect a quarter of a right angle.

4. If  $A$  be the vertex of an isosceles triangle  $ABC$ , and  $BA$  be produced to  $D$ , so that  $AD$  is equal to  $BA$ , and  $DC$  be drawn: then  $BCD$  is a right angle.

5. A straight line drawn at right angles to  $BC$  the base of an isosceles triangle  $ABC$  cuts  $AB$  in  $D$  and  $CA$  produced in  $E$ : prove that  $AED$  is an isosceles triangle.

6. Construct a right-angled triangle having given the hypotenuse and the sum of the sides.

7. The line joining the right angle of a right-angled triangle to the middle point of the hypotenuse is equal to half the hypotenuse.

8. The locus of the vertices of all right-angled triangles which have a common hypotenuse is a circle.

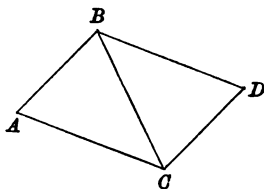
## PROPOSITION 33.

*If two sides of a convex quadrilateral be equal and parallel, the other sides are equal and parallel.*

Let  $ABDC$  be a quadrilateral, in which the sides  $AB$ ,  $CD$  are equal and parallel :

it is required to prove that the sides  $AC$ ,  $BD$  are equal and parallel.

CONSTRUCTION. Draw one of the diagonals  $BC$ .



PROOF. Because  $AB$  is parallel to  $CD$ ,  
and  $BC$  meets them,  
the alternate angles  $ABC$ ,  $BCD$  are equal. (Prop. 29.)

Because in the triangles  $ABC$ ,  $DCB$ ,  
 $AB$  is equal to  $DC$ ,  
and  $BC$  to  $CB$ ,  
and the angle  $ABC$  to the angle  $DCB$ ,  
the triangles are equal in all respects ; (Prop. 4.)  
therefore the angle  $ACB$  is equal to the angle  $DBC$ ,  
and  $CA$  to  $BD$ .

And because the straight line  $BC$  meets the two straight lines  $AC$ ,  $BD$ , and makes the alternate angles  $ACB$ ,  $CBD$  equal to one another,

$AC$  is parallel to  $BD$ . (Prop. 27.)

And it was proved to be equal to it.

Wherefore, *if two sides &c.*

## EXERCISES.

1. Draw a straight line so that the part intercepted between two given straight lines is equal to one given straight line and parallel to another.

2. If a quadrilateral have two of its opposite sides parallel, and the two others equal but not parallel, any two of its opposite angles are together equal to two right angles.

3. If a straight line which joins the extremities of two equal straight lines, not parallel, make the angles on the same side of it equal to each other, the straight line which joins the other extremities will be parallel to the first.

4. If from  $D$  any point in the base  $BC$  of an isosceles triangle  $ABC$ ,  $DE$ ,  $DF$  be drawn perpendicular to the sides, then the sum of  $DE$ ,  $DF$  is constant.

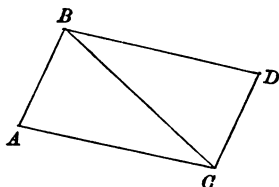
## PROPOSITION 34.

*Opposite sides of a parallelogram are equal, and opposite angles are equal; and a diagonal of a parallelogram bisects its area.*

Let  $ACDB$  be a parallelogram, of which  $BC$  is a diagonal:

it is required to prove that (1) opposite sides are equal,  $AB$  to  $CD$ , and  $AC$  to  $BD$ ;

(2) opposite angles are equal,  $BAC$  to  $BDC$  and  $ABD$  to  $ACD$ ;  
and (3) the diagonal  $BC$  bisects the area of the parallelogram.



**PROOF.** Because  $AB$  is parallel to  $CD$ , and  $BC$  meets them,

the alternate angles  $ABC$ ,  $BCD$  are equal. (Prop. 29.)

And because  $AC$  is parallel to  $BD$ , and  $BC$  meets them,  
the alternate angles  $ACB$ ,  $CBD$  are equal. (Prop. 29.)

Now because in the two triangles  $ABC$ ,  $DCB$ ,

the angle  $ABC$  is equal to the angle  $DCB$ ,

and the angle  $BCA$  to the angle  $CBD$ ,

and the side  $BC$  adjacent to the equal angles in each is common to both,

the triangles are equal in all respects. (Prop. 26, Part 1.)

Therefore  $AB$  is equal to  $DC$ ,  $AC$  equal to  $DB$ ,

and the angle  $BAC$  equal to the angle  $CDB$ .

And because the angle  $ABC$  is equal to the angle  $DCB$ ,

and the angle  $CBD$  to the angle  $BCA$ ,

the whole angle  $ABD$  is equal to the whole angle  $DCA$ .

And the angle  $BAC$  has been proved to be equal to the angle  $CDB$ .

Therefore in the parallelogram  $AD$  (1) opposite sides are equal and (2) opposite angles are equal.

Again, it has been proved that the triangles  $ABC$ ,  $DCB$  are equal in all respects: therefore (3) the diagonal  $BC$  bisects the area of the parallelogram  $AD$ .

Wherefore, *opposite sides &c.*

COROLLARY 1. *All the sides of a square are equal.*

COROLLARY 2. *The angles made by a pair of straight lines are equal to the angles made by any pair of straight lines parallel to them.*

A parallelogram  $ABCD$  is often spoken of as the parallelogram  $AC$ , or the parallelogram  $BD$ , or more simply as  $AC$  or  $BD$ , when there is no danger of confusion with the diagonal  $AC$  or with the diagonal  $BD$ .

### EXERCISES.

1. Prove that, if the diagonals of a quadrilateral bisect one another, the quadrilateral is a parallelogram. Prove also the converse.

2. If two sides of a quadrilateral be parallel and the other two equal but not parallel, the diagonals are equal.

3. If in a quadrilateral the diagonals be equal and two sides be parallel, the other sides are equal.

4. Find in a side of a triangle the point from which straight lines drawn parallel to the other sides of the triangle and terminated by them are equal.

5. Prove that every straight line which bisects the area of a parallelogram must pass through the intersection of its diagonals.

6. Construct a triangle whose angles shall be equal to those of a given triangle, and whose area shall be four times the area of the given triangle.

7.  $ABCD$  is a parallelogram having the side  $AD$  double of  $AB$ : the side  $AB$  is produced both ways to  $E$  and  $F$  till each produced part equals  $AB$ , and straight lines are drawn from  $C$  and  $D$  to  $E$  and  $F$  so as to cross within the figure: shew that they will meet at right angles.

8. If  $O$  be any point within a parallelogram  $ABCD$ , the sum of the triangles  $OAB$ ,  $OCD$  is half the parallelogram.

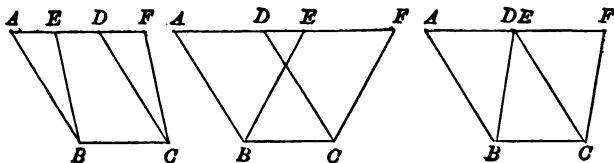
9. Divide a given straight line into  $n$  equal parts, where  $n$  is a whole number.

## PROPOSITION 35.

*Two parallelograms, which have one side common and the sides opposite to the common side in a straight line, are equal in area.*

Let  $ABCD$ ,  $EBCF$  be two parallelograms, which have a common side  $BC$ , and the sides  $AD$ ,  $EF$  in a straight line:

it is required to prove that  $ABCD$ ,  $EBCF$  are equal in area.



PROOF. Because  $ABCD$  is a parallelogram,  
 $AB$  is equal to  $DC$ , (Prop. 34.)

and because  $EBCF$  is a parallelogram,

$BE$  is equal to  $CF$ ;

and because  $AB$  is parallel to  $DC$ ,

and  $BE$  to  $CF$ ,

the angle  $ABE$  is equal to the angle  $DCF$ .

(Prop. 34, Coroll. 2.)

And because in the triangles  $ABE$ ,  $DCF$ ,

$AB$  is equal to  $DC$ ,

and  $BE$  to  $CF$ ,

and the angle  $ABE$  to the angle  $DCF$ ,

the triangles are equal in all respects. (Prop. 4.)

Take from the area  $ABCF$ , the equal areas  $FDC$ ,  $EAB$ ;

then the remainders are equal,

that is, the parallelograms  $ABCD$ ,  $EBCF$  are equal in area.

Wherefore, *two parallelograms &c.*

The propositions in the remaining part of the First Book of Euclid and those in the Second Book relate chiefly to cases of equality of the areas of two figures.

The test of equality to which we have hitherto always appealed has been that of the possibility of shifting one figure so that it exactly coincides with the other. In this case the figures are equal in all respects, but we say that two figures are *equal in area* when, when it is possible to shift all the parts of the area of one figure, so that they together exactly fit the area of the second figure.

It will be observed that this is the test made use of in Proposition 35.

For the future we shall often, when there is no danger of ambiguity, speak of the equality of two figures when we mean only equality of area, and we shall often speak of a figure when we mean only the area of the figure.

#### EXERCISES.

1. Construct a rectangle equal to a given parallelogram.
2. Construct a rhombus equal to a given parallelogram.
3. Construct a parallelogram to be equal to a given parallelogram in area and to have its sides equal to two given straight lines. Is this always possible?



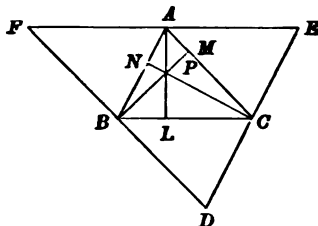


## ADDITIONAL PROPOSITION.

The straight lines, drawn from the vertices of a triangle perpendicular to the opposite sides, meet in a point\*.

Let  $ABC$  be a triangle, and  $AL$ ,  $BM$ ,  $CN$  be drawn perpendicular to  $BC$ ,  $CA$ ,  $AB$  respectively.

Draw the straight lines  $FAE$ ,  $DBF$ ,  $ECD$  parallel to  $BC$ ,  $CA$ ,  $AB$  respectively.



Because  $BE$  is a parallelogram,

$AE$  is equal to  $BC$ ;

(Prop. 34.)

and because  $CF$  is a parallelogram,

$FA$  is equal to  $BC$ ;

therefore  $FA$  is equal to  $AE$ .

Again, because  $AL$  meets the parallels  $FAE$ ,  $BLC$ ,

the angle  $FAL$  is equal to the angle  $ALC$ . (Prop. 29.)

But the angle  $ALC$  is a right angle;

therefore the angle  $FAL$  is a right angle.

Therefore  $AL$  is the straight line drawn at right angles to  $FE$  at its middle point.

Similarly it can be proved that  $BM$ ,  $CN$  are the straight lines drawn at right angles to  $FD$ ,  $DE$  at their middle points.

Now  $AL$ ,  $BM$ ,  $CN$  the straight lines drawn at right angles to the sides of the triangle  $DEF$  at their middle points meet in a point.

(Add. Prop., page 58.)

Therefore  $AL$ ,  $BM$ ,  $CN$  the straight lines drawn from the vertices of the triangle  $ABC$  perpendicular to the opposite sides meet in a point.

## EXERCISES.

1. Construct a parallelogram to be equal to a given parallelogram and to have one of its sides in a given straight line.
2. Construct a parallelogram to be equal to a given parallelogram and to have two of its sides in two given straight lines.

\* This point is often called the orthocentre of the triangle.

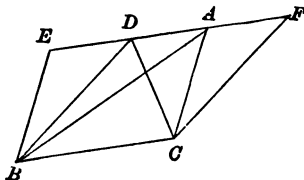
## PROPOSITION 37.

*Two triangles, which have one side common and the angular points opposite to the common side on a straight line parallel to it, are equal.*

Let  $ABC$ ,  $DBC$  be two triangles, which have a common side  $BC$ , and their angular points  $A$ ,  $D$  on a straight line  $AD$  parallel to  $BC$  :

it is required to prove that the triangles  $ABC$ ,  $DBC$  are equal.

CONSTRUCTION. Through  $B$  draw  $BE$  parallel to  $CA$ , and through  $C$  draw  $CF$  parallel to  $BD$ , (Prop. 31.) meeting  $AD$  (produced if necessary) in  $E$  and  $F$ .



PROOF. Because the parallelograms  $EBCE$ ,  $DBCF$  have a common side  $BC$  and the sides  $EA$ ,  $DF$  in a straight line,

$EBCE$  is equal to  $DBCF$ . (Prop. 35.)

And because the diagonal  $AB$  bisects the parallelogram  $EBCE$ ,

the triangle  $ABC$  is half of  $EBCE$ ; (Prop. 34.)

and because the diagonal  $DC$  bisects the parallelogram  $DBCF$ ,

the triangle  $DBC$  is half of  $DBCF$ .

Now the halves of equals are equal.

Therefore the triangles  $ABC$ ,  $DBC$  are equal.

Wherefore, *two triangles &c.*

## EXERCISES.

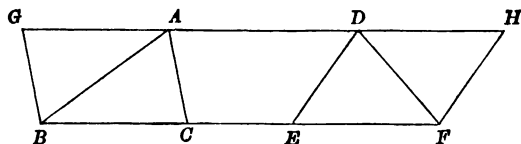
1. If  $P$  be a point within a parallelogram  $ABCD$ , the difference of the triangles  $PAB$ ,  $PAD$  is equal to the triangle  $PAC$ .
2. If  $P$  be a point outside a parallelogram  $ABCD$ , the sum of the triangles  $PAB$ ,  $PAD$  is equal to the triangle  $PAC$ .
3.  $AB$  and  $ECD$  are two parallel straight lines:  $BF$ ,  $DF$  are drawn parallel to  $AD$ ,  $AE$  respectively: prove that the triangles  $ABC$ ,  $DEF$  are equal to one another.
4.  $ABC$  is a given triangle: construct a triangle of equal area, having  $AB$  for base and its vertex in a given straight line.
5. Points  $A$ ,  $B$ ,  $C$  are taken, one on each of three parallel straight lines:  $BC$ ,  $CA$ ,  $AB$  meet the lines through  $A$ ,  $B$ ,  $C$  respectively in  $a$ ,  $b$ ,  $c$ : prove that each of the triangles  $ABC$ ,  $Abc$ ,  $Bca$ ,  $Cab$ , is equal to half the triangle  $abc$ .

## PROPOSITION 38.

*Two triangles, which have two sides equal and in a straight line and also have the angular points opposite to the equal sides on a straight line parallel to it, are equal.*

Let  $ABC$ ,  $DEF$  be two triangles, which have their sides  $BC$ ,  $EF$  equal and in a straight line, and their angular points  $A$ ,  $D$ , on a straight line  $AD$  parallel to  $BF$ : it is required to prove that the triangles  $ABC$ ,  $DEF$  are equal.

CONSTRUCTION. Through  $B$  draw  $BG$  parallel to  $CA$ , and through  $F$  draw  $FH$  parallel to  $ED$ , meeting  $AD$  (produced if necessary) in  $G$  and  $H$ .



PROOF. Because the parallelograms  $GBCA$ ,  $DEFH$  have their sides  $BC$ ,  $EF$  equal and in a straight line, and also their sides  $GA$ ,  $DH$  in a straight line, they are equal to one another. (Prop. 36.)

Because the diagonal  $AB$  bisects the parallelogram  $GBCA$ , the triangle  $ABC$  is half of  $GBCA$ ; (Prop. 34.) and because the diagonal  $DF$  bisects the parallelogram  $DEFH$ ,

the triangle  $DEF$  is half of  $DEFH$ .

Now the halves of equals are equal;

therefore the triangles  $ABC$ ,  $DEF$  are equal.

Wherefore, two triangles &c.

COROLLARY. *Two triangles, which have two sides equal and in a straight line and also have the angular points opposite to the equal sides coincident, are equal.*

## EXERCISES.

1.  $ABCD$  is a parallelogram; from any point  $P$  in the diagonal  $BD$  the straight lines  $PA, PC$  are drawn. Shew that the triangles  $PAB$  and  $PCB$  are equal in area.

2. The three sides of a triangle are bisected, and the points of bisection are joined; prove that the triangle is divided into four triangles, which are all equal to one another.

3. If the sides  $BC, CA, AB$  of a triangle  $ABC$  be produced to  $A', B', C'$  respectively, so that  $CA' = BC, AB' = CA, AB = BC'$ , prove that the area of the triangle  $A'B'C'$  is seven times that of the triangle  $ABC$ .

4. Make a triangle such as to be equal to a given parallelogram, and to have one of its angles equal to a given angle.

5. If the sides  $AB, BC, CA$  of a triangle  $ABC$  be respectively bisected in  $c, a, b$ , and  $Aa, Cc$  intersect in  $P$ : then  $BPb$  is a straight line.

6. The sides  $AB, AC$  of a triangle are bisected in  $D, E$ ;  $CD, BE$  intersect in  $F$ . Prove that the triangle  $BFC$  is equal to the quadrilateral  $ADFE$ .

7. If  $AB, PQRS, CD$  be three parallel straight lines and  $P, Q, R, S$  be situate on  $AC, AD, BC, BD$  respectively, then  $PQ$  is equal to  $RS$ , and  $PR$  to  $QS$ .

8.  $A', B', C'$  are the middle points of the sides of the triangle  $ABC$ , and through  $A, B, C$  are drawn three parallel straight lines meeting  $B'C', C'A', A'B'$  in  $a, b, c$  respectively; prove that the triangle  $abc$  is half the triangle  $ABC$  and that  $bc$  passes through  $A$ ,  $ca$  through  $B$ ,  $ab$  through  $C$ .

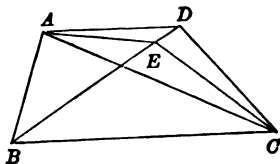
## PROPOSITION 39.

*If two equal triangles have a common side and lie on the same side of it, the angular points opposite to the common side lie on a straight line parallel to it.*

Let  $ABC$ ,  $DBC$  be two equal triangles, which have a common side  $BC$ , and lie on the same side of  $BC$  :

it is required to prove that the angular points  $A$ ,  $D$  opposite to the side  $BC$  lie on a straight line parallel to  $BC$ .

CONSTRUCTION. Draw  $AD$ , and in  $BD$  or  $BD$  produced take any point  $E$  other than  $D$ , and draw  $AE$ ,  $EC$ .



PROOF. Because the triangle  $DBC$  is not equal to the triangle  $EBC$ ,

and the triangle  $ABC$  is equal to the triangle  $DBC$ ,  
the triangle  $ABC$  is not equal to the triangle  $EBC$ .

If  $AE$  were parallel to  $BC$ ,  
the triangle  $ABC$  would be equal to the triangle  $EBC$  ;  
(Prop. 37.)

but they are not equal ;  
therefore  $AE$  is not parallel to  $BC$ .

But it is possible to draw a straight line through  $A$  parallel  
to  $BC$  ;  
(Prop. 31.)

therefore  $AD$  is parallel to  $BC$ .

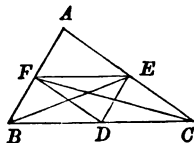
Wherefore, if two equal triangles &c.

## ADDITIONAL PROPOSITION.

*Each side of a triangle is double of the straight line joining the middle points of the other sides and is parallel to it.*

Let  $ABC$  be a triangle,  $D, E, F$  the middle points of the sides  $BC, CA, AB$ .

Draw  $BE, CF, EF, FD, DE$ .



Because the two triangles  $BFC, AFC$  have their sides  $BF, AF$  equal and in a straight line, and the point  $C$  common,

the triangles are equal; (Prop. 38, Coroll.)

therefore the triangle  $BFC$  is half of the triangle  $ABC$ .

Similarly it can be proved that the triangle  $BEC$  is half of the triangle  $ABC$ .

Therefore the triangle  $BFC$  is equal to the triangle  $BEC$ .

Next because the triangles  $BFC, BEC$  are equal and have a common side  $BC$ , the straight line  $FE$  joining their vertices is parallel to  $BC$ . (Prop. 39.)

Similarly it can be proved that  $DF$  is parallel to  $CA$  and  $ED$  to  $AB$ .

Again because  $BFED$  is a parallelogram,  
 $BD$  is equal to  $FE$ .

(Prop. 34.)

And because  $DFEC$  is a parallelogram,  
 $DC$  is equal to  $FE$ :

therefore  $BC$  is double of  $FE$ .

## EXERCISES.

1. The middle points of the sides of any quadrilateral are the angular points of a parallelogram.

2. Of equal triangles on the same base, the isosceles triangle has the least perimeter.

3. Two triangles of equal area stand on the same base and on opposite sides: shew that the straight line joining their vertices is bisected by the base or the base produced.

4. The triangle  $ABC$  is double of the triangle  $EBC$ : shew that, if  $AE, BC$  produced if necessary meet in  $D$ , then  $AE$  is equal to  $ED$ .

5. If the straight lines joining the middle points of two of the sides of a triangle to the opposite vertices be equal, the triangle is isosceles.

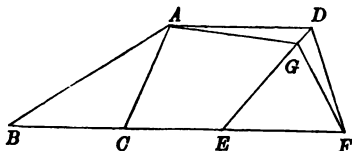
## PROPOSITION 40.

*If two equal triangles have two sides equal and in a straight line, and if the triangles lie on the same side of this line, the angular points opposite to the equal sides lie on a straight line parallel to the first straight line.*

Let  $ABC$ ,  $DEF$  be two equal triangles, which have equal sides  $BC$ ,  $EF$  in a straight line and lie on the same side of  $BF$ :

it is required to prove that the angular points  $A$ ,  $D$  opposite to  $BC$ ,  $EF$  lie on a straight line parallel to  $BF$ .

CONSTRUCTION. Draw  $AD$ , and in  $ED$  or  $ED$  produced take any point  $G$  other than  $D$ , and draw  $AG$ ,  $GF$ .



PROOF. Because the triangle  $DEF$  is not equal to the triangle  $GEF$ ,

and the triangle  $ABC$  is equal to the triangle  $DEF$ ,  
the triangle  $ABC$  is not equal to the triangle  $GEF$ .

If  $AG$  were parallel to  $BF$ ,  
the triangle  $ABC$  would be equal to the triangle  $GEF$ ;  
(Prop. 38.)

but they are not equal;

therefore  $AG$  is not parallel to  $BF$ .

But it is possible to draw a straight line through  $A$  parallel to  $BF$ ;  
(Prop. 31.)

therefore  $AD$  is parallel to  $BF$ .

Wherefore, if two equal triangles &c.



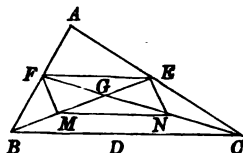
## ADDITIONAL PROPOSITION.

*The straight lines joining the vertices of a triangle to the middle points of the opposite sides meet in a point\* which is for each line the point of trisection further from the vertex.*

Let  $ABC$  be a triangle, and  $D, E, F$  be the middle points of the sides  $BC, CA, AB$ .

Draw  $BE, CF$  and let them intersect in  $G$ .

Bisect  $BG, CG$  in  $M, N$ , and draw  $FM, MN, NE$ .



In the triangle  $ABC$ ,  
 $BC$  is double of  $FE$  and is parallel to it. (Add. Prop., page 101.)

In the triangle  $GBC$ ,  
 $BC$  is double of  $MN$  and is parallel to it.

Therefore  $FE$  is equal and parallel to  $MN$ . (Prop. 30.)

Therefore  $FMNE$  is a parallelogram. (Prop. 33.)

Now the diagonals of a parallelogram bisect each other.

(Exercise 1, page 91.)

Therefore  $GE$  is equal to  $GM$ , which is equal to  $MB$ .

Therefore  $BG$  is double of  $GE$ .

Similarly  $CG$  is double of  $GF$ .

Similarly it can be proved that  $AD$  passes through  $G$ , and that  $AG$  is double of  $GD$ .

## EXERCISES.

1. A point  $P$  is taken within a quadrilateral  $ABCD$ : prove that, if the sum of the areas of the triangles  $PAB, PCD$  be independent of the position of  $P$ ,  $ABCD$  is a parallelogram.

2. The locus of a point  $P$  such that the sum of the areas of the two triangles  $PAB, PBC$  is constant, is a straight line parallel to  $AC$ .

3.  $AB, CD$  are two given straight lines: the locus of a point  $P$  such that the sum of the two triangles  $PAB, PCD$  is constant, is a straight line.

4. Trisect a given straight line.

\* This point is often called the centre of gravity or the centroid of the triangle.

## PROPOSITION 41.

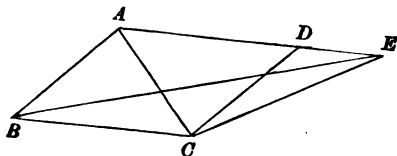
*If a parallelogram and a triangle have a common side, and the angular point of the triangle opposite to the common side lie on the same straight line as the opposite side of the parallelogram, the parallelogram is double of the triangle.*

Let  $ABCD$  be a parallelogram and  $EBC$  be a triangle which have a common side  $BC$ , and let the angular point  $E$  of the triangle lie in the same straight line as the side  $AD$  of the parallelogram :

it is required to prove that the parallelogram  $ABCD$  is double of the triangle  $EBC$ .

CONSTRUCTION.

Draw  $AC$ .



PROOF. Because the triangles  $ABC$ ,  $EBC$ , have a common side  $BC$ , and  $AE$  is parallel to  $BC$ ,  
the triangles  $ABC$ ,  $EBC$  are equal. (Prop. 37.)

And because the diagonal  $AC$  bisects the parallelogram  $ABCD$ ,

the parallelogram  $ABCD$  is double of the triangle  $ABC$ .

(Prop. 34.)

Therefore the parallelogram  $ABCD$  is double of the triangle  $EBC$ .

Wherefore, *if a parallelogram &c.*

## EXERCISES.

1.  $ABCD$  is a parallelogram; from  $D$  draw any straight line  $DFG$  meeting  $BC$  at  $F$  and  $AB$  produced at  $G$ ; draw  $AF$  and  $CG$ : shew that the triangles  $ABF$ ,  $CFG$  are equal.

2. If  $P$  be a point in the side  $AB$ , and  $Q$  a point in the side  $DC$  of a parallelogram  $ABCD$ , then the triangles  $PCD$ ,  $QAB$  are equal in area.

3. The area of any convex quadrilateral is double that of the parallelogram whose vertices are the middle points of the sides of the quadrilateral.

4. The sides  $BC$ ,  $CA$ ,  $AB$  of a triangle  $ABC$  are trisected in the points  $D$ ,  $d$ ;  $E$ ,  $e$ ;  $F$ ,  $f$  respectively: prove that the area of the hexagon  $DdEeFf$  is two-thirds that of the triangle  $ABC$ .

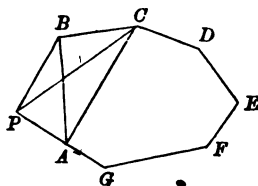
## PROPOSITION 41 A.

*To construct a triangle equal to a given rectilinear figure.*

Let  $ABCDEFG$  be the given rectilinear figure:  
it is required to construct a triangle equal to  $ABCDEFG$ .

CONSTRUCTION. Draw one of the diagonals  $AC$  such that with two adjacent sides of the figure  $AB, BC$  it forms a triangle  $ABC$ .

Through the vertex  $B$  draw  $BP$  parallel to  $CA$ , to meet  $GA$  produced in  $P$ . Draw  $PC$ .



PROOF. Because the triangles  $PAC, BAC$  have a common side  $AC$ , and their angular points  $P, B$  on a straight line parallel to  $AC$ :

the two triangles  $PAC, BAC$  are equal. (Prop. 37.)

Add to each the figure  $ACDEFG$ ;  
then the figure  $PCDEFG$  is equal to the figure  $ABCDEFG$ .

Now the sides of the figure  $PCDEFG$  are fewer by one than the sides of the figure  $ABCDEFG$ ; therefore by continued application of this process we can construct a series of figures all equal to the given figure, the sides of each figure being fewer by one than the sides of the figure last preceding.

We shall thus ultimately obtain a triangle equal to the given rectilinear figure.

It will be seen that by the method adopted in Proposition 41 A a triangle can be constructed equal to a given rectilineal figure of 4 sides by using the process once, to a figure of 5 sides by using it twice, and to a figure of  $n$  sides by using it  $n-3$  times.

## EXERCISES.

1. On one side of a given triangle construct an isosceles triangle equal to the given triangle.

2. On one side of a given quadrilateral construct a rectangle equal to the quadrilateral.

3. Construct a triangle equal in area to a given convex five-sided figure  $ABCDE$ :  $AB$  is to be one side of the triangle and  $AE$  the direction of one of the other sides.

4. Bisect a given (1) parallelogram, (2) triangle, (3) quadrilateral by a straight line drawn through a given point in one side of the figure.

5.  $ABCD$  is a given quadrilateral: construct a quadrilateral of equal area, having  $AB$  for one side, and another side on a given straight line parallel to  $AB$ .

6.  $ABCD$  is a given quadrilateral: construct a triangle, whose base shall be in the same straight line as  $AB$ , its vertex at a given point  $P$  in  $CD$ , and its area equal to that of the given quadrilateral.

## PROPOSITION 42.

*To construct a parallelogram equal to a given triangle, and having an angle equal to a given angle.*

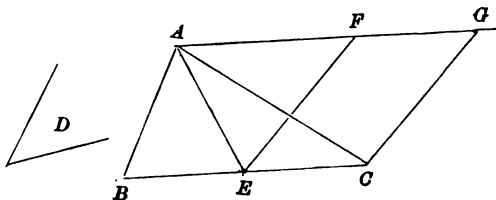
Let  $ABC$  be the given triangle, and  $D$  the given angle: it is required to construct a parallelogram equal to  $ABC$ , and having an angle equal to  $D$ .

CONSTRUCTION. Bisect  $BC$  at  $E$ :

draw  $AE$ ,

and from the point  $E$ , in the straight line  $EC$ , draw  $EF$  making the angle  $CEF$  equal to the angle  $D$ ; (Prop. 23.)

through  $A$  draw  $AFG$  parallel to  $EC$  meeting  $EF$  in  $F$ , and through  $C$  draw  $CG$  parallel to  $EF$  meeting  $AFG$  in  $G$ : then  $FECG$  is a parallelogram constructed as required.



PROOF. Because the opposite sides of the quadrilateral  $FECG$  are parallel,

$FECG$  is a parallelogram.

Because the triangles  $ABE$ ,  $AEC$  have the sides  $BE$ ,  $EC$  equal and in a straight line, and the angular point  $A$  common, the triangle  $ABE$  is equal to the triangle  $AEC$ ;

(Prop. 38, Coroll.)

therefore the triangle  $ABC$  is double of the triangle  $AEC$ .

Because the parallelogram  $FECG$  and the triangle  $AEC$  have a common side  $EC$  and the point  $A$  lies on the same straight line as the side  $FG$ ,

(Prop. 41.)

the parallelogram  $FECG$  is double of the triangle  $AEC$ .

Therefore the parallelogram  $FECG$  is equal to the triangle  $ABC$ , and it has an angle  $CEF$  equal to the given angle  $D$ .

Wherefore a parallelogram  $FECG$  has been constructed equal to the given triangle  $ABC$ , and having an angle  $CEF$  equal to the given angle  $D$ .

**EXERCISES:**

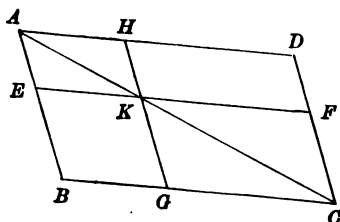
1. On one side of a given triangle construct a rectangle equal to the triangle.
2. On one side of a given triangle construct a rhombus equal to the triangle. Is this always possible?
3. On one side of a given triangle as diagonal construct a rhombus equal to the triangle.

## PROPOSITION 43.

*Complements of parallelograms about a diagonal of a parallelogram, are equal.*

Let  $ABCD$  be a parallelogram, of which  $AC$  is a diagonal; and  $EH$ ,  $GF$  are parallelograms about  $AC$ ; and  $KB$ ,  $KD$  the complements: (See note on page 111.)

it is required to prove that  $KB$  is equal to  $KD$ .



PROOF. Because  $BD$  is a parallelogram, and  $AC$  a diagonal,

the triangle  $ABC$  is equal to the triangle  $ADC$ . (Prop. 34.)

Now the triangle  $ABC$  is equal to the two triangles  $AEK$ ,  $KGC$  and the parallelogram  $KB$ ;

and the triangle  $ADC$  is equal to the two triangles  $AHK$ ,  $KFC$  and the parallelogram  $KD$ .

Therefore the two triangles  $AEK$ ,  $KGC$  and the parallelogram  $KB$  are together equal to the two triangles  $AHK$ ,  $KFC$  and the parallelogram  $KD$ .

Again, because  $EH$  is a parallelogram and  $AK$  a diagonal, the triangle  $AEK$  is equal to the triangle  $AHK$ ;

(Prop. 34.)

and because  $GF$  is a parallelogram, and  $KC$  a diagonal, the triangle  $KGC$  is equal to the triangle  $KFC$ . (Prop. 34.)

Therefore taking away equals from equals, the remainder, the complement  $KB$ , is equal to the remainder, the complement  $KD$ .

Wherefore, *complements of parallelograms &c.*



If through a point  $K$  on a diagonal  $AC$  of a parallelogram  $ABCD$ , straight lines  $HKG$ ,  $EKF$  be drawn parallel to the sides  $AB$ ,  $BC$  respectively to meet the sides  $AD$ ,  $BC$ ,  $AB$ ,  $DC$  in  $H$ ,  $G$ ,  $E$ ,  $F$  respectively; then  $EH$ ,  $GF$  are called parallelograms about the diagonal  $AC$ , and the parallelograms  $EG$ ,  $FH$  are called complements of these parallelograms.

## EXERCISES.

1. Prove that in the diagram of Proposition 43, the following are pairs of equal triangles:  $ABK$  and  $ADK$ ;  $AEC$  and  $AHC$ ;  $AKG$  and  $AKF$ .
2. The diagonals of parallelograms about a diagonal of a parallelogram are parallel.
3. Parallelograms about a diagonal of a square are squares.
4. If  $ABCD$ ,  $AEFG$  be two squares so placed that the angles at  $A$  coincide, then  $A$ ,  $F$ ,  $C$  lie on a straight line.
5. If through  $E$  a point within a parallelogram  $ABCD$  straight lines be drawn parallel to  $AB$ ,  $BC$ , and the parallelograms  $AE$ ,  $EC$  be equal, the point  $E$  lies in the diagonal  $BD$ .

## PROPOSITION 44.

*To construct a parallelogram equal to a given parallelogram, having an angle equal to an angle of the given parallelogram, and having a side equal to a given straight line.*

Let  $ABCD$  be the given parallelogram, and  $EF$  the given straight line :

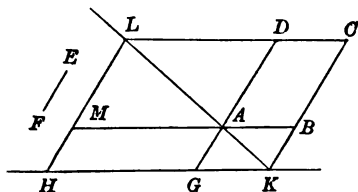
it is required to construct a parallelogram equal to  $ABCD$ , having an angle equal to the angle  $BAD$ , and having a side equal to  $EF$ .

CONSTRUCTION. Produce  $DA$  to  $G$ , and make  $AG$  equal to  $EF$ . (Prop. 3.)

Through  $G$  draw  $HGK$  parallel to  $AB$  meeting  $CB$  produced in  $K$ . (Prop. 31.)

Draw  $KA$  and produce it to meet  $CD$  produced in  $L$ , and through  $L$  draw  $LMH$  parallel to  $DAG$  to meet  $BA$  produced in  $M$  and  $HGK$  in  $H$  :

then  $MAGH$  is a parallelogram constructed as required.



PROOF. Because  $LCKH$  is a parallelogram,  $KL$  a diagonal, and  $MG$ ,  $BD$  complements of parallelograms about the diagonal  $KL$ ,

$MG$  is equal to  $BD$ . (Prop. 43.)

Again, because the straight lines  $BAM$ ,  $DAG$  intersect at  $A$ , the angle  $MAG$  is equal to the vertically opposite angle  $BAD$ . (Prop. 15.)

And  $AG$  is equal to  $EF$ . (Constr.)

Wherefore, a parallelogram  $MAGH$  has been constructed equal to the given parallelogram  $ABCD$ , having an angle  $MAG$  equal to the angle  $BAD$  and having a side  $AG$  equal to the given straight line  $EF$ .

**EXERCISES.**

1. On a given straight line construct a rectangle equal to a given rectangle.
2. On a given straight line construct a rhombus equal to a given triangle. Is this always possible?
3. Construct a rectangle equal to the sum of two given rectangles.
4. Construct a rectangle equal to the difference of two given rectangles.

## PROPOSITION 45.

*To construct a parallelogram equal to a given rectilineal figure, having a side equal to a given straight line, and having an angle equal to a given angle.*

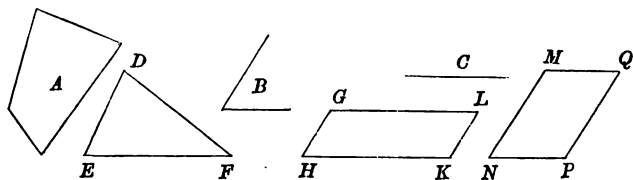
Let  $A$  be the given rectilineal figure,  $B$  the given angle, and  $C$  the given straight line:

it is required to construct a parallelogram equal to the figure  $A$ , having an angle equal to the angle  $B$ , and having a side equal to  $C$ .

CONSTRUCTION. Construct the triangle  $DEF$  equal to the figure  $A$ . (Prop. 41 A.)

Construct the parallelogram  $GHLK$  equal to the triangle  $DEF$ , having the angle  $GHK$  equal to the angle  $B$ . (Prop. 42.)

Construct the parallelogram  $MNPQ$  equal to the parallelogram  $GHLK$ , having an angle  $MNP$  equal to the angle  $GHK$ , and having the side  $MN$  equal to  $C$ . (Prop. 43.)



PROOF. Because the triangle  $DEF$  is equal to the figure  $A$ , (Constr.)

and the parallelogram  $GK$  is equal to the triangle  $DEF$ , (Constr.)

and the parallelogram  $MP$  is equal to the parallelogram  $GK$ , (Constr.)

the parallelogram  $MP$  is equal to the figure  $A$ .

Because the angle  $GHK$  is equal to the angle  $B$ , (Constr.)

and the angle  $MNP$  is equal to the angle  $GHK$ , (Constr.)

the angle  $MNP$  is equal to the angle  $B$ .

And  $MN$  is equal to  $C$ . (Constr.)

Wherefore a parallelogram  $MNPQ$  has been constructed equal to the given rectilineal figure  $A$ , having the side  $MN$  equal to the given straight line  $C$ , and having the angle  $MNP$  equal to the given angle  $B$ .

## EXERCISES.

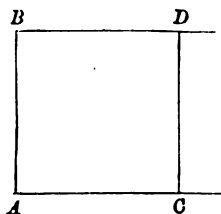
1. On a given straight line as diagonal, construct a rhombus equal to a given triangle.
2. Construct a right-angled triangle, having given the hypotenuse and the perpendicular from the right angle on it. (See Exercise 8, page 87.)
3. Construct a rectangle equal to a given rectangle, and having a diagonal equal to a given straight line.
4. Construct a rectangle equal to the sum of two given triangles.

## PROPOSITION 46.

*On a given straight line to construct a square.*

Let  $AB$  be the given straight line:  
it is required to construct a square on  $AB$ .

CONSTRUCTION. From the point  $A$  draw  $AC$  at right angles to  $AB$ ; (Prop. 11.)  
and make  $AC$  equal to  $AB$ ; (Prop. 3.)  
through  $B$  draw  $BD$  parallel to  $AC$ , (Prop. 31.)  
and through  $C$  draw  $CD$  parallel to  $AB$  meeting  $BD$  in  $D$ :  
then  $ABDC$  is a square constructed as required.



PROOF. Because  $CD$  is parallel to  $AB$ ,  
and  $BD$  to  $AC$ ,  
the figure  $ABDC$  is a parallelogram. (Def. 18.)

Again the angle  $CAB$  is a right angle;  
therefore the parallelogram  $ABDC$  is a rectangle. (Def. 19.)

Again, the adjacent sides  $AC$ ,  $AB$  are equal; (Constr.)  
therefore the rectangle  $ABDC$  is a square. (Def. 20.)

Wherefore,  $ABDC$  is a square constructed on the given straight line  $AB$ .

## EXERCISES.

1. If two squares be equal in area, their sides are equal.
2. The squares on two equal straight lines are equal in all respects.
3. If in the sides  $AB, BC, CD, DA$  of a square points  $E, F, G, H$  be taken so that  $AE, BF, CG, DH$  are equal: then  $EFGH$  is a square.
4. If the diagonals of a quadrilateral be equal and bisect each other at right angles, the quadrilateral is a square.
5. On the sides  $AC, BC$  of a triangle  $ABC$ , squares  $ACDE, BCFG$  are constructed: shew that the straight lines  $AF$  and  $BD$  are equal.
6. Construct a square so that one side shall be in a given straight line and two other sides shall pass through two given points.
7. Construct a square so that two opposite sides shall pass through two given points, and its diagonals intersect at a third given point.
8. Prove that the straight line, bisecting the right angle of a right-angled triangle, passes through the intersection of the diagonals of the square constructed on the outer side of the hypotenuse.

## PROPOSITION 47.

*In a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other sides.*

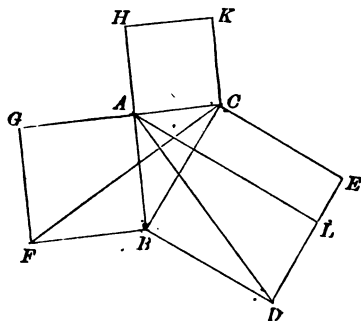
Let  $ABC$  be a right-angled triangle, having the right angle  $BAC$ :

it is required to prove that the square on  $BC$  is equal to the sum of the squares on  $BA$ ,  $AC$ .

CONSTRUCTION. On  $BC$  on the side away from  $A$  construct the square  $BDEC$ , (Prop. 46.)  
and similarly on  $BA$ ,  $AC$  construct the squares  $BAGF$ ,  $ACKH$ ;

through  $A$  draw  $AL$  parallel to  $BD$  meeting  $DE$  in  $L$ ;  
(Prop. 31.)

and draw  $AD$ ,  $FC$ .



PROOF. Because each of the angles  $BAC$ ,  $BAG$  is a right angle, the two straight lines  $AC$ ,  $AG$ , on opposite sides of  $AB$ , make with it at  $A$  the adjacent angles together equal to two right angles;

therefore  $CA$  is in the same straight line with  $AG$ .

(Prop. 14.)

The angle  $DBC$  is equal to the angle  $FBA$ ,  
for each of them is a right angle. (Prop. 10 A.)



Add to each of these equals the angle  $ABC$ ;  
then the angle  $DBA$  is equal to the angle  $FBC$ .

And because in the triangles  $ABD$ ,  $FBC$ ,

$AB$  is equal to  $FB$  and  $BD$  to  $BC$ ;

and the angle  $ABD$  is equal to the angle  $FBC$ ;

the triangles  $ABD$ ,  $FBC$  are equal in all respects. (Prop. 4.)

Because the parallelogram  $BL$  and the triangle  $ABD$  have  
a common side  $BD$  and  $A$  is in the same straight line  
as the side of  $BL$  opposite to  $BD$ , (Prop. 41.)

the parallelogram  $BL$  is double of the triangle  $ABD$ .

And because the square  $GB$  and the triangle  $FBC$  have a  
common side  $FB$ , and  $C$  is in the same straight line as  
the side of  $GB$  opposite to  $FB$ , (Prop. 41.)

the square  $GB$  is double of the triangle  $FBC$ .

Now the doubles of equals are equal.

Therefore the parallelogram  $BL$  is equal to the square  $GB$ .

Similarly it can be proved that the parallelogram  $CL$  is  
equal to the square  $HC$ .

Therefore the whole square  $BDEC$ , which is the sum of the  
rectangles  $BL$ ,  $CL$ , is equal to the sum of the two squares  
 $GB$ ,  $HC$ .

And the square  $BDEC$  is constructed on  $BC$ , and the squares  
 $GB$ ,  $HC$  on  $BA$ ,  $AC$ .

Therefore the square on the side  $BC$  is equal to the sum of  
the squares on the sides  $BA$ ,  $AC$ .

Wherefore, in a right-angled triangle &c.

#### EXERCISES.

1. Construct a square equal to the difference of two given squares.
2. The diagonals of a quadrilateral intersect at right angles. Prove that the sum of the squares on one pair of opposite sides is equal to the sum of the squares on the other pair.
3. If  $O$  be the point of intersection of the perpendiculars drawn from the angles of a triangle upon the opposite sides, the squares on  $OA$  and  $BC$  are together equal to the squares on  $OB$  and  $CA$ , and also to the squares on  $OC$  and  $AB$ .
4. Divide a given straight line into two parts so that the sum of the squares on the parts may be equal to a given square.
5. Divide a given straight line so that the difference of the squares on the parts is equal to a given square.

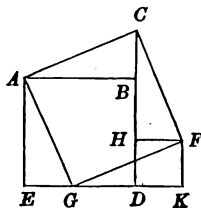
The proof of the theorem "the square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other sides," which we have given in the text of the 47th proposition, is attributed to Euclid.

Tradition however says that the first person to discover a proof of the truth of the theorem was Pythagoras, a Greek philosopher who lived between 570 and 500 B.C. The theorem is in consequence often quoted as **the Theorem of Pythagoras**. What was the nature of Pythagoras' proof is not known.

The theorem is one of great importance and a large number of proofs of its truth have been discovered. It is advisable that the student should be made acquainted with some proofs besides the one given in the text.

We have made a selection of five proofs of the theorem : in each case not attempting to give the complete proof, but merely giving hints of the line of argument to be used, and leaving the student to develop it more fully.

**PROOF I.** Take a right-angled triangle  $ABC$ , and on the side  $AB$  away from  $C$  construct the square  $ABDE$ , and on the hypotenuse  $AC$  on the same side as  $B$  construct the square  $ACFG$ . From  $F$  draw  $FH$  perpendicular to  $BD$ , and  $FK$  perpendicular to  $ED$  produced.



It may be proved that

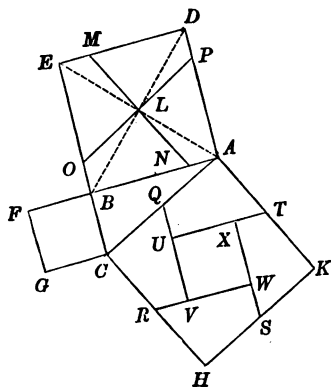
- (1)  $CBD$  is a straight line,
- (2)  $G$  lies in  $DE$ ,
- (3) the triangles  $ABC$ ,  $AEG$ ,  $CHF$ ,  $GKF$  are all equal,
- (4)  $HK$  is a square and equal to the square on  $BC$ .

PROOF II. Take a right-angled triangle  $ABC$  and on the sides  $AB, BC, CA$  away from  $C, A, B$  construct the squares  $BADE, CBFG, ACHK$ .

Through  $L$  the intersection of the diagonals  $AE, BD$  of the square on the larger side  $AB$ , draw  $MLN$  perpendicular to  $CA$  and  $OLP$  parallel to  $CA$ .

Take  $Q, R, S, T$  the middle points of the sides  $AC, CH, HK, KA$  of the square on the hypotenuse.

Through  $Q, S$  draw  $QUV, SWX$  parallel to  $BC$ , and through  $R, T$  draw  $RVW, TXU$  parallel to  $AB$ .



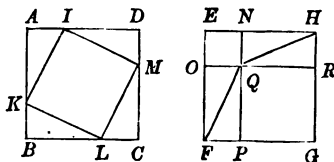
It may be proved that

- (1) all the quadrilaterals  $LMEO, LOBN, LNAP, LPDM, AQU, CRVQ, HSWR, KTXS$  are equal to one another,
- (2) the quadrilateral  $UVWX$  is a square,
- (3) the squares  $CF, UW$  are equal.

PROOF III. Take two equal squares  $ABCD$ ,  $EFGH$ .

Take any point  $I$  in  $AD$ , and measure off  $BK$ ,  $CL$ ,  $DM$ ,  $EN$ ,  $EO$  each equal to  $AI$ .

Draw  $IK$ ,  $KL$ ,  $LM$ ,  $MI$ ; through  $N$  draw  $NQP$  parallel to  $EF$ , and through  $O$  draw  $OQR$  parallel to  $EH$ . Draw  $QF$ ,  $QH$ .



It may be proved that

(1) the square  $ABCD$  is divided into one square  $IKLM$  and four equal right-angled triangles,

(2) the square  $EFGH$  is divided into two squares  $EOQN$ ,  $QPGR$  and four equal triangles,

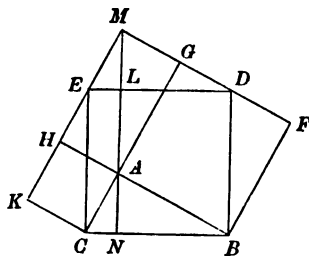
(3) all the triangles are equal to each other,

(4) the square  $IL$  is equal to the sum of the squares  $ON$ ,  $PR$ ,

(5) the three squares  $IL$ ,  $ON$ ,  $PR$  are squares on the hypotenuse and on the sides of one or other of the equal triangles.

PROOF IV. Take a right-angled triangle  $ABC$  and on the hypotenuse  $BC$  on the same side as  $A$  construct the square  $BCED$ , and on the sides  $CA$ ,  $AB$  away from  $B$ ,  $C$  construct the squares  $CAHK$ ,  $ABFG$ .

Through  $A$  draw  $MLAN$  perpendicular to  $BC$ , and produce  $FG$ ,  $KH$  to meet  $MLAN$ .



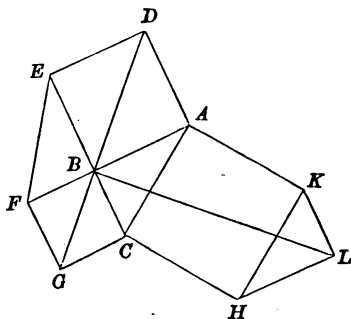
It may be proved that

- (1)  $D$  lies in  $FG$ ,
- (2)  $E$  lies in  $KH$  produced,
- (3) the rectangle  $BL$ , and the square  $AF$  are each equal to the parallelogram  $AD$ ,
- (4) the rectangle  $CL$  and the square  $AK$  are each equal to the parallelogram  $AE$ .

**PROOF V.** Take a right-angled triangle  $ABC$ : on the sides  $AB$ ,  $BC$ ,  $CA$  away from  $C$ ,  $A$ ,  $B$  construct the squares  $BADE$ ,  $CBFG$ ,  $ACHK$ .

On  $HK$  construct a triangle  $HLK$  equal in all respects to the triangle  $ABC$  having  $HL$  parallel to  $AB$ , and  $KL$  to  $CB$ .

Draw  $FE$ ,  $GB$ ,  $BD$ ,  $BL$ .



It may be proved that

- (1)  $GBD$  is a straight line.
- (2) the triangles  $FBE$ ,  $CBA$  are equal,
- (3) all the quadrilaterals  $GFED$ ,  $GCAD$ ,  $BCHL$ ,  $LKAB$  are equal to one another.

## PROPOSITION 48.

*If the square on one side of a triangle be equal to the sum of the squares on the other sides, the angle contained by these two sides is a right angle.*

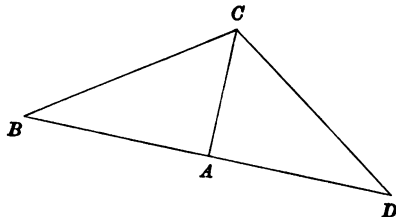
Let the square on  $BC$ , one of the sides of the triangle  $ABC$ , be equal to the sum of the squares on the other sides  $BA$ ,  $AC$ :

it is required to prove that the angle  $BAC$  is a right angle.

CONSTRUCTION. From the point  $A$  draw  $AD$  at right angles to  $AC$ ; (Prop. 11.)

and make  $AD$  equal to  $BA$ ; (Prop. 3.)

and draw  $DC$ .



PROOF. Because  $DA$  is equal to  $BA$ ,  
the square on  $DA$  is equal to the square on  $BA$ .

To each of these equals add the square on  $AC$ ;  
then the sum of the squares on  $DA$ ,  $AC$  is equal to the  
sum of the squares on  $BA$ ,  $AC$ .

Now because the angle  $DAC$  is a right angle,  
the square on  $DC$  is equal to the sum of the squares on  
 $DA$ ,  $AC$ . (Prop. 47.)

And the square on  $BC$  is equal to the sum of the squares  
on  $BA$ ,  $AC$ . (Hypothesis.)

Therefore the square on  $DC$  is equal to the square on  $BC$ ,  
and  $DC$  is equal to  $BC$ .

And because in the triangles  $DAC$ ,  $BAC$ ,

$DA$  is equal to  $BA$ ,

$AC$  to  $AC$ ,

and  $CD$  to  $CB$ ,

the triangles are equal in all respects; (Prop. 8.)

therefore the angle  $DAC$  is equal to the angle  $BAC$ .

Now  $DAC$  is a right angle ; (Constr.)  
therefore  $BAC$  is a right angle.

Wherefore, *if the square &c.*

### EXERCISES.

1. If the difference of the squares on two sides of a triangle be equal to the square on the third side, the triangle is right angled.

2. The locus of a point, such that the difference of the squares on its distances from two given points is equal to a given square, is a straight line.

3. Prove by means of Proposition 48 that the straight lines, drawn from the vertices of a triangle perpendicular to the opposite sides, meet in a point.

4. If the sum of the squares on two opposite sides of a quadrilateral be equal to the sum of the squares on the other two sides, the diagonals of the quadrilateral intersect at right angles.

5.  $ABCD$  is a quadrilateral such that, if any point  $P$  be joined to  $A, B, C, D$ , the sum of the squares on  $PA, PC$  is equal to the sum of the squares on  $PB, PD$ : prove that  $ABCD$  is a rectangle.

## MISCELLANEOUS EXERCISES.

1. How many diagonals can be drawn through the same vertex in (1) a quadrilateral, (2) a hexagon, (3) a polygon of  $n$  sides?
2. How many different diagonals can be drawn to (1) a quadrilateral, (2) a hexagon, (3) a polygon of  $n$  sides?
3. If two straight lines bisect each other at right angles, any point in either of them is equidistant from the extremities of the other.
4. A straight line drawn bisecting the angle contained by two equal sides of a triangle bisects the third side at right angles.
5. Two isosceles triangles  $CAB$ ,  $DAB$  are on the same base  $AB$ : shew that the triangles  $ACD$ ,  $BCD$  are equal in all respects.
6. Prove by the method of superposition that, if two isosceles triangles have the same vertical angle, their bases are parallel.
7. If  $ABC$ ,  $DBC$  be two triangles equal in all respects on opposite sides of  $BC$ , then  $AD$  is perpendicular to  $BC$  and is bisected by it.
8. The angle  $BAC$  of a triangle  $ABC$  is bisected by a straight line which meets  $BC$  in  $D$ , and from  $AB$  on  $AB$  produced  $AE$  is cut off equal to  $AC$ : prove that  $DE$  is equal to  $DC$ .
9. If two circles whose radii are equal cut in  $A$ ,  $B$  and if the line joining their centres when produced meet the circumference in  $C$ ,  $D$ , prove that  $ACBD$  is a rhombus.
10. Prove by the method of superposition that, if the opposite angles of a quadrilateral be equal and one pair of opposite sides be equal, the other sides are equal.
11. Prove by the method of superposition that, if a quadrilateral has two pairs of adjacent angles equal, it has one pair of opposite sides equal.
12. Two adjacent sides of a quadrilateral are equal, and the two angles which they form with the other sides are together equal to the angle between the other sides. Prove that one diagonal of the quadrilateral is equal to a side.
13. In a triangle  $BAC$  is the greatest angle. Prove that, if a point  $D$  be taken in  $AB$ , and a point  $E$  in  $AC$ ,  $DE$  is less than  $BC$ .
14. If  $AD$  be drawn perpendicular from the vertex  $A$  to the opposite side  $BC$  of a triangle  $ABC$  in which  $AC$  is greater than  $AB$ , then  $DC$  is greater than  $BD$ , and the angle  $DAC$  is greater than the angle  $BAD$ .
15. How many different triangles can be formed by taking three lines out of six lines whose lengths are 2, 3, 4, 5, 6, 7 inches respectively?



16. Find the point the sum of whose distances from the four angular points of a convex quadrilateral is a minimum.

17.  $AB$  is a given finite straight line. From  $C$  the middle point of  $AB$ ,  $CD$  is drawn in any direction and of any length. Prove that  $AD$ ,  $BD$  together are greater than twice  $CD$ .

18. From a given point draw a straight line making equal angles with two given intersecting straight lines. How many such lines can be drawn?

19. In a given straight line find a point equally distant from two given straight lines. In what case is a solution impossible?

20. How many equalities must be given between the sides and the angles of (1) two quadrilaterals, (2) two hexagons, (3) two polygons of  $n$  sides, before the conclusion can be drawn that the figures are equal in all respects?

21. In the triangle  $ABC$  the angles at  $B$  and  $C$  are equal;  $m$  and  $l$  are points in  $AC$  produced and on  $AB$  respectively, and  $lm$  is joined cutting  $BC$  in  $O$ . Prove that, if the sum of  $Al$  and  $Am$  be double  $AB$ , then  $BO$  is greater than  $CO$ .

22. The side  $BC$  of a triangle  $ABC$  is produced to a point  $D$ ; the angle  $ACB$  is bisected by the straight line  $CE$  which meets  $AB$  at  $E$ . A straight line is drawn through  $E$  parallel to  $BC$ , meeting  $AC$  at  $F$ , and the straight line bisecting the exterior angle  $ACD$  at  $G$ . Shew that  $EF$  is equal to  $FG$ .

23. A straight line drawn at right angles to  $BC$  the base of an isosceles triangle  $ABC$  cuts the side  $AB$  at  $D$  and  $CA$  produced at  $E$ : shew that  $AED$  is an isosceles triangle.

24. If in the base of a triangle  $ABC$ , there be taken any two points  $P$  and  $Q$  equidistant from the extremities of the base, and if through each of the points  $P$ ,  $Q$  two straight lines be drawn parallel to  $AB$ ,  $AC$ , so as to form two parallelograms having  $PA$ ,  $QA$  for diagonals; these two parallelograms are equal in area.

25. Find a point equidistant from each of three straight lines in a plane which do not coincide in direction with the sides of any triangle that can be drawn in the plane. Is the construction required in this last problem always possible?

26. From a point  $P$  outside an angle  $BAC$  draw a straight line cutting the straight lines  $AB$ ,  $AC$  in points  $D$  and  $E$  such that  $PD$  may be equal to  $DE$ .

27. If one acute angle of a triangle be double of another, the triangle can be divided into two isosceles triangles.

28. If in a triangle  $ABC$ ,  $ACB$  be a right angle, and the angle  $CAB$  be double the angle  $ABC$ , then  $AB$  is double  $AC$ .

29.  $P$  is a point in the side  $CD$  of a square  $ABCD$  such that  $AP$  is equal to the sum of  $PC$  and  $CB$  and  $Q$  is the middle point of  $CD$ . Prove that the angle  $BAP$  is twice the angle  $QAD$ .

30.  $AOB$ ,  $COD$  are two indefinite straight lines intersecting each other in the point  $O$ , and  $P$  is a given point in the plane of these lines. It is required to draw through the point  $P$  a straight line  $PXY$  cutting  $AB$  in  $X$  and  $CD$  in  $Y$ , in such a manner that  $OX$  may be equal to  $OY$ . Can this problem be solved in more than one way?

31. Construct an equilateral triangle one of whose angular points is given and the other two lie one on each of two given straight lines.

32. The sides  $AB$ ,  $AC$  of a triangle are bisected in  $D$ ,  $E$ , and  $BE$ ,  $CD$  are produced to  $F$ ,  $G$ , so that  $EF$  is equal to  $BE$ , and  $DG$  to  $CD$ : prove that  $FAG$  is a straight line.

33. In a plane triangle an angle is acute, right or obtuse, according as the straight line joining the angle to the middle point of the opposite side is greater than, equal to, or less than half that side.

34. The difference of the angles at the base of a triangle is double the angle between the perpendicular to the base, and the bisector of the vertical angle.

35.  $AC$  is the longest side of the triangle  $ABC$ . Find in  $AC$  a point  $D$  such that the angle  $ADB$  shall be equal to twice the angle  $ACB$ .

36.  $ABCD$  is a quadrilateral: the bisectors of the angles  $ABD$ ,  $ACD$  meet at  $F$ : prove that the angle  $BFC$  is half the sum of the angles  $BAC$ ,  $BDC$ .

37. Prove that, if points  $L$ ,  $M$ ,  $N$  be taken in the sides  $BC$ ,  $CA$ ,  $AB$  of a triangle  $ABC$ , such that the triangles  $ANM$ ,  $NBL$ ,  $MLC$  are equiangular to each other, they are equiangular also to the triangles  $ABC$  and  $LMN$ .

38. If one angle at the base of a triangle be double the other, the less side is equal to the sum or the difference of the segments of the base made by the perpendicular from the vertex, according as the first angle is greater or less than a right angle.

39. Any convex pentagon  $ABCDE$  has each of its angular points joined to the non-contiguous points and a star-shaped figure  $ACEBD$  is formed: prove that the sum of the angles of the figure  $ACEBD$  is two right angles.

40. In the figure of Proposition 1, if  $AB$  produced both ways meet the circles again in  $D$  and  $E$  and  $CD$ ,  $CE$  be drawn,  $CDE$  will be an isosceles triangle having one angle four times each of the other angles.

41. From the extremities of the base of an isosceles triangle straight lines are drawn perpendicular to the sides; shew that the angles made by them with the base are each equal to half the vertical angle.

42. The sides  $AB$ ,  $AC$  of a given triangle  $ABC$  are bisected at the points  $E$ ,  $F$ ; a perpendicular is drawn from  $A$  to the opposite side, meeting it at  $D$ . Shew that the angle  $FDE$  is equal to the angle  $BAC$ .

43.  $AB$ ,  $AC$  are two given straight lines, and  $P$  is a given point in the former: it is required to draw through  $P$  a straight line to meet  $AC$  at  $Q$ , so that the angle  $APQ$  may be three times the angle  $AQP$ .

44. Construct a right-angled triangle having given the hypotenuse and the difference of the sides.

45. From a given point it is required to draw to two parallel straight lines, two equal straight lines at right angles to each other.

46. Construct a triangle of given perimeter, having its angles equal to those of a given triangle.

47. Given one angle, and the opposite side, and the sum of the other sides, construct the triangle.

48. If two triangles on the same side of a common base have their sides terminated in opposite extremities of the base equal, the line joining the vertices will be parallel to the common base.

49. If an exterior angle of a triangle be bisected, and also one of the interior and opposite angles, the angle contained by the bisecting lines is equal to half the other interior and opposite angle.

50. If the perpendicular drawn from one of the equal angles of an isosceles triangle upon the opposite side divide the angle into two parts, one of which is double of the other, the vertical angle of the triangle is either one half or four-fifths of a right angle.

51. The sides  $AB$ ,  $AC$  of a triangle  $ABC$  are produced to  $E$ ,  $F$ , and the angles  $CBE$ ,  $BCF$  are bisected by the straight lines  $BD$ ,  $CD$ : prove that the angle  $BDC$  is half the sum of the angles  $ABC$ ,  $ACB$ .

52. In a triangle  $ABC$  points  $D$ ,  $E$  are taken in  $AC$  such that  $AD$  is equal to  $AB$ , and that  $CE$  is equal to  $CB$ , and a point  $F$  is taken in  $AB$  such that  $BF$  is equal to  $BC$ : prove that the angle  $EBD$  is equal to the angle  $BCF$ .

53.  $ABC$  is a triangle, and from a point  $D$  in  $AB$  the straight line  $DEF$  is drawn meeting  $BC$  in  $E$  and  $AC$  produced in  $F$ ; shew that the angles between the straight lines bisecting the angles  $ABE$ ,  $ADE$  are equal to the angles between the straight lines bisecting the angles  $ACE$ ,  $AFE$ .

54. Find a point in a given straight line such that its distance from a given point is double of its distance from a given straight line through the given point.

55. Construct an equilateral triangle, having given its altitude.

56. The straight line bisecting the exterior angle at the vertex  $A$  of a triangle  $ABC$  will meet the base  $BC$  produced beyond  $B$ , or beyond  $C$ , according as  $AC$  is greater or less than  $AB$ .

57. If from any point within an isosceles triangle perpendiculars be let fall on the base and the sides, the sum of these perpendiculars is less than the altitude of the triangle, if the vertical angle be less than the angle of an equilateral triangle.

58.  $ABC$  is a triangle right-angled at  $A$ , and on  $AB$ ,  $AC$  are described two equilateral triangles  $ABD$ ,  $ACE$  (both on the outside or both on the inside), and  $DB$  and  $EC$  or those produced meet in  $F$ ; shew that  $A$  is the ortho-centre of the triangle  $DEF$ .

59. If the angle between two adjacent sides of a parallelogram be increased, while their lengths do not alter, the diagonal through their point of intersection will diminish.

60. If straight lines be drawn from the angles of any parallelogram perpendicular to any straight line which is outside the parallelogram, the sum of those from one pair of opposite angles is equal to the sum of those from the other pair of opposite angles.

61. If a six-sided plane rectilineal figure have its opposite sides equal and parallel, the three straight lines joining the opposite angles will meet at a point.

62. The vertical angle  $CAB$  of a triangle  $ABC$  is bisected by  $AD$ , and  $BE$ ,  $CF$  are drawn perpendicular to  $AD$ : prove that the middle point of  $BC$  is equidistant from  $E$  and  $F$ .

63. Find in a side of a triangle a point such that the sum of two straight lines drawn from the point parallel to the other sides and terminated by them is equal to a given straight line.

64. If the angular points of one parallelogram lie on the sides of another parallelogram, the diagonals of both parallelograms pass through the same point.

65. If the straight line joining two opposite angles of a parallelogram bisect the angles, the parallelogram is a rhombus.

66. Draw a straight line through a given point such that the part of it intercepted between two given parallel straight lines may be of given length.

67. Bisect a parallelogram by a straight line drawn through a given point within it.

68. Shew that the four triangles into which a parallelogram is divided by its diagonals are equal in area.

69. Straight lines bisecting two adjacent angles of a parallelogram intersect at right angles.

70. Straight lines bisecting two opposite angles of a parallelogram are either parallel or coincident.

71. Find a point such that the perpendiculars let fall from it on two given straight lines shall be respectively equal to two given straight lines. How many such points are there?

72.  $ABCD$  is a quadrilateral having  $BC$  parallel to  $AD$ ,  $E$  is the middle point of  $DC$ ; shew that the triangle  $AEB$  is half the quadrilateral.

73. If the sides of a triangle be trisected and straight lines be drawn through the points of section adjacent to each angle so as to form another triangle, this is equal to the original triangle in all respects.

74. If two opposite sides of a parallelogram be bisected and two straight lines be drawn from the points of bisection to two opposite angles, the two lines trisect the diagonal which passes through the other two angular points.

75.  $BAC$  is a right-angled triangle,  $A$  being the right angle.  $ACDE$ ,  $BCFG$  are squares on  $AC$  and  $BC$ .  $AC$  produced meets  $DF$  in  $K$ . Prove that  $DF$  is bisected in  $K$ , and that  $AB$  is double of  $CK$ .

76. In a triangle  $ABC$  on  $AC$ ,  $BC$  on the sides of them away from  $B$ ,  $A$ , squares  $ACDE$ ,  $BCFG$  are constructed; prove that, if  $AC$  produced bisect  $DF$ ,  $BAC$  is a right angle.

77. A straight line  $PQ$  drawn parallel to the diagonal  $AC$  of a parallelogram  $ABCD$  meets  $AB$  in  $P$  and  $BC$  in  $Q$ ; shew that the other diagonal  $BD$  bisects the quadrilateral  $BPDQ$ .

78. If the opposite angles of a quadrilateral be equal, the opposite sides are equal.

79. If in two parallelograms  $ABCD$ ,  $EFGH$ ,  $AB$  be equal to  $EF$ ,  $BC$  to  $FG$ , and the angle  $ABC$  to the angle  $EFG$ , then the parallelograms are equal in all respects.

80. If the straight line  $AB$  be bisected in  $C$ , and  $AD$  and  $BE$  be drawn at right angles to  $AB$ , and  $AD$  be taken equal to  $AC$ , and  $DE$  be drawn at right angles to  $DC$ ,  $DE$  is double of  $DC$ .

81.  $ABC$  is a given triangle; construct a triangle of equal area, having its vertex at a given point in  $BC$  and its base in the same straight line as  $AB$ .

82. In the right-angled triangle  $ABC$ , the sides  $AB$ ,  $AC$  which contain the right angle are equal. A second right-angled triangle is constructed having the sides containing the right angle together equal to  $AB$ ,  $AC$ , but not equal to one another. Prove that this triangle is less than the triangle  $ABC$ .

83. If two triangles have two sides of the one equal to two sides of the other, and the sum of the two angles contained by these sides equal to two right angles, the triangles are equal in area.

84. If a triangle be described having two of its sides equal to the diagonals of any quadrilateral, and the included angle equal to either of the angles between these diagonals, then the area of the triangle is equal to the area of the quadrilateral.

85.  $ABC$  is a triangle; find the locus of a point  $O$  such that the sum of the areas  $OAB$ ,  $OBC$ ,  $OCA$  is constant and greater than the area of  $ABC$ .

86. Any point  $P$  is joined to  $O$  the middle point of  $AD$ . On  $AP$ ,  $DP$  squares  $APQQ'$ ,  $DPRE'$  are described on the sides remote from  $D$ ,  $A$  respectively. Prove that  $QR$  is double of and perpendicular to  $OP$ .

87.  $ABCD$  is a parallelogram;  $E$  the point of bisection of  $AB$ ; prove that  $AC$ ,  $DE$  being joined will each pass through a point of trisection of the other.

88. In every quadrilateral the intersection of the straight lines which join the middle points of opposite sides is the middle point of the straight line which joins the middle points of the diagonals.

89. The line joining the middle points of the diagonals of the quadrilateral  $ABCD$  meets  $AD$  and  $BC$  in  $E$  and  $F$ . Shew that the triangles  $EBE$ ,  $FAD$  are each half the quadrilateral.

90.  $PQR$  is a straight line parallel and equal to the base  $BC$  of a triangle meeting the sides in  $P$  and  $Q$ . Shew that the triangles  $BPQ$ ,  $AQR$  are equal.

91. Two straight lines  $AB$  and  $CD$  intersect at  $E$ , and the triangle  $AEC$  is equal to the triangle  $BED$ ; shew that  $BC$  is parallel to  $AD$ .

92. Construct the smallest triangle, which has a given vertical angle, and whose base passes through a given point.

93. In the base  $AC$  of a triangle take any point  $D$ ; bisect  $AD$ ,  $DC$ ,  $AB$ ,  $BC$  at the points  $E$ ,  $F$ ,  $G$ ,  $H$  respectively; shew that  $EG$  is equal and parallel to  $FH$ .

94. Given the middle points of the sides of a triangle, construct the triangle.

95.  $ABC$  is a triangle. The side  $CA$  is bisected in  $D$  and  $E$  is the point of trisection of the side  $BC$  which is nearer to  $B$ . Shew that the line joining  $A$  to  $E$  bisects the line joining  $B$  to  $D$ .

96. Shew how by means of Prop. 40 to produce a finite straight line beyond an obstacle which cannot be passed through directly.

97.  $BAC$  is a fixed angle of a triangle, and (i) the sum, (ii) the difference of the sides  $AB$ ,  $AC$  is given; shew that in either case the locus of the middle point of  $BC$  is a straight line.

98. Through a fixed point  $O$  two straight lines  $OPQ$ ,  $OP'Q'$  are drawn meeting two fixed parallel straight lines. Prove that, if  $PQ'$  and  $P'Q$  meet in  $R$ , the locus of  $R$  is a straight line, and that  $OR$  bisects  $PP'$  and  $QQ'$ .

99.  $ABC$  is a triangle and on the side  $BC$  a parallelogram  $BDEC$  is described, and the parallelogram whose adjacent sides are  $AD$ ,  $AB$  is completed and also that whose adjacent sides are  $AE$ ,  $AC$ ; shew that the sum or the difference of the two latter parallelograms is equal to the first.

100.  $ABC$  is a triangle;  $ADF$ ,  $CFE$  are the perpendiculars let fall from  $A$  and  $C$ , one on the internal bisector of the angle  $B$  and the other on the external bisector: the area of the rectangle  $BDFE$  is equal to that of the triangle.

101. If on the sides  $AB$ ,  $BC$ ,  $CA$  of any triangle, squares  $ABEF$ ,  $BCGH$ ,  $CAKL$  be constructed, and  $EH$ ,  $GL$ ,  $KF$  be drawn, then all the triangles  $ABC$ ,  $BEH$ ,  $CGL$ ,  $AKF$  are equal to one another.

102. Construct a square, which shall have two adjacent sides passing through two given points, and the intersection of diagonals at a third given point.

103.  $ABC$  is a triangle right-angled at  $A$  and  $K$  is the corner of the square on  $AB$  opposite to  $A$  and  $H$  the corner of the square on  $AC$  opposite to  $A$ . If  $AB$  be produced to  $D$ , so that  $AD$  is equal to  $CK$  and  $AC$  be produced to  $E$ , so that  $AE$  is equal to  $BH$ , then  $CD$  is equal to  $BE$ .

104. Upon  $BC$ ,  $CA$ ,  $AB$ , sides of the triangle  $ABC$ , perpendiculars are drawn from a point  $D$ , meeting the sides, or the sides produced, in  $E$ ,  $F$ ,  $G$  respectively. Prove that the sum of the squares on  $AG$ ,  $BE$ ,  $CF$  is equal to the sum of the squares on  $BG$ ,  $CE$ ,  $AF$ .

105. Divide a given straight line into two parts such that the square on one of them may be double the square on the other.

## BOOK II.

### DEFINITIONS.

**DEFINITION 1.** *A rectangle is said to be contained by the two sides which contain any one of its angles.*

The expression, that a rectangle is *contained* by two straight lines  $AB$ ,  $BC$  is of course a faulty one, as the area of the rectangle is contained by the *four* sides of the figure. The expression must be considered merely as an abbreviated form of the statement that the rectangle has for two of its adjacent sides the two straight lines  $AB$ ,  $BC$ .

It can easily be proved that, if two adjacent sides of one rectangle be equal to two adjacent sides of another rectangle, the two rectangles are equal in all respects.

(See Exercise 79, page 131.)

A rectangle, two of whose adjacent sides are equal to two straight lines  $AB$ ,  $CD$ , is therefore often said to be the rectangle contained by  $AB$ ,  $CD$ ; such a rectangle is often denominated simply **the rectangle  $AB$ ,  $CD$** .

It is clear that the rectangle  $AB$ ,  $CD$  is the same as the rectangle  $CD$ ,  $AB$ .

Also it may be noticed that, if  $AB$  be equal to  $CD$ , the rectangle  $AB$ ,  $CD$  is equal to the square on  $AB$ , or to the square on  $CD$ .



In Arithmetic or in Algebra, if we wish to represent a given length, we take a definite length, for instance an inch, as a *unit of length* and we express the given length by the *number* of units of length contained in it.

In the same way, if we wish to represent a given area, we take a definite area, for instance a square inch, as a *unit of area*, and we express the given area by the *number* of units of area contained in it.

It is easily seen that, if a rectangle have 2 inches in one side and 3 inches in an adjacent side, its area consists of  $2 \times 3$  or 6 squares, each square having one inch as its side, and similarly that, if a rectangle have  $m$  units of length in one side and  $n$  units of length in an adjacent side, its area consists of  $mn$  squares, each square having a unit of length as its side.

Thus in Arithmetic or in Algebra the area of a rectangle is represented by the *product of the numbers*, which represent the lengths of two adjacent sides.

If the rectangle be a square, its area is represented by the *square of the number*, which represents the length of a side.

In consequence of this connection between Algebra and Geometry, there is a certain correspondence between the theorems and problems of the Second Book of Euclid, and theorems and problems in Algebra.

A short statement of a corresponding proposition in Algebra is given as a note to each Proposition, in which statement each straight line is represented by a corresponding letter, and each area by a corresponding product.

PROPOSITION I.

If there be two straight lines, one of which is divided into any two parts, the rectangle contained by the two straight lines is equal to the sum of the rectangles contained by the undivided line and each of the parts of the divided line\*.

Let  $AB$  and  $CD$  be two straight lines; and let  $CD$  be divided into any two parts at the point  $E$ :  
It is required to prove that the rectangle contained by  $AB$  and  $CD$  is equal to the sum of the rectangles contained by  $AB$  and  $CE$ , and by  $AB$  and  $ED$ .

From the point  $C$  draw  $CF$  at right angles to  $CD$ ; (I. Prop. 11.)  
and make  $CE$  equal to  $AB$ ; (I. Prop. 3.)  
through  $E$  draw  $EF$  parallel to  $CF$ ; (I. Prop. 31.)  
through  $F$  draw  $FA$  parallel to  $CD$ ; meeting  $AB$  in  $A$ .



Since  $CE$  is equal to  $AB$ ,  $EA$  is a parallel to  $CF$ ; (Constr.)  
and the figure has two right angles;  
therefore each of the figures is a rectangle;  
therefore the rectangle  $AB$  is equal to the sum of the rectangles  $ABCE$  and  $EFED$ .

The algebraical equivalent of this theorem is the equation  
 $a \cdot b = a \cdot c + a \cdot d$ .

But  $CK$  is contained by  $AB, CD$ ,  
for it is contained by  $CG, CD$ , and  $CG$  is equal to  $AB$ .

And  $CH$  is contained by  $AB, CE$ ,  
for it is contained by  $CG, CE$ , and  $CG$  is equal to  $AB$ .

And  $EK$  is contained by  $AB, ED$ ,  
for it is contained by  $EH, ED$ , and  $EH$  is equal to  $CG$ ,  
which is equal to  $AB$ .

Therefore the rectangle contained by  $AB, CD$  is equal to  
the sum of the rectangles contained by  $AB, CE$ , and by  
 $AB, ED$ .

Wherefore, *if there be two straight lines &c.*

**COROLLARY 1.** *If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the sum of the rectangles contained by the undivided line and each of the parts of the divided line.*

If  $CD$  be divided into three parts at the points  $E, F$ :  
the rectangle  $AB, CD$  is equal to the sum of the rectangles  
 $AB, CE$ , and  $AB, ED$ : (Prop. 1.)

and the rectangle  $AB, ED$  is equal to the sum of the rectangles  
 $AB, EF$  and  $AB, FD$ :

therefore the rectangle  $AB, CD$  is equal to the sum of the  
rectangles  $AB, CE$ ;  $AB, EF$  and  $AB, FD$ .

And so on for any number of points of division.

**COROLLARY 2.** *If there be two straight lines, one of which is divided into any two parts, the rectangle contained by the undivided line and one of the parts of the divided line is equal to the difference of the rectangles contained by the undivided line and the whole of the divided line and by the undivided line and the remaining part of the divided line.*

#### EXERCISES.

1. If  $A, B, C, D$  be four points in order in a straight line, then the sum of the rectangles  $AB, CD$ , and  $AD, BC$  is equal to the rectangle  $AC, BD$ .

2. If there be two straight lines, each of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the sum of the rectangles contained by each of the parts of the first line and each of the parts of the second line.

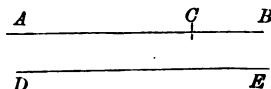
## PROPOSITION 2.

*If a straight line be divided into any two parts, the square on the whole line is equal to the sum of the rectangles contained by the whole and each of the parts\*.*

Let the straight line  $AB$  be divided into any two parts at the point  $C$ :

it is required to prove that the square on  $AB$  is equal to the sum of the rectangles contained by  $AB$ ,  $AC$  and by  $AB$ ,  $CB$ .

CONSTRUCTION. Take  $DE$  a straight line equal to  $AB$ .



PROOF. The rectangle  $DE$ ,  $AB$  is equal to the sum of the rectangles  $DE$ ,  $AC$  and  $DE$ ,  $CB$ . (Prop. 1.)

Because  $DE$  is equal to  $AB$ ,  
 the rectangle  $DE$ ,  $AB$  is equal to the square on  $AB$ ,  
 the rectangle  $DE$ ,  $AC$  to the rectangle  $AB$ ,  $AC$ ,  
 and the rectangle  $DE$ ,  $CB$  to the rectangle  $AB$ ,  $CB$ :  
 therefore the square on  $AB$  is equal to the sum of the rectangles  $AB$ ,  $AC$ , and  $AB$ ,  $CB$ .

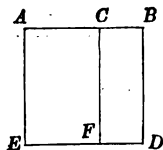
Wherefore, if a straight line &c.

\* The algebraical equivalent of this theorem is the equation

$$(a+b)^2 = (a+b)a + (a+b)b.$$

*Outline of Alternative Proof.*

On  $AB$  construct the square  $AEDB$ ,  
and draw  $CF$  parallel to  $AE$  to meet  $ED$  in  $F$ .



It may be proved that

- (1)  $AF$  is the rectangle  $AB, AC$ ,  
and (2)  $CD$  is the rectangle  $AB, CB$ ,  
and hence that the square on  $AB$  is equal to the sum of  
the rectangles  $AB, AC$  and  $AB, CB$ .

## EXERCISES.

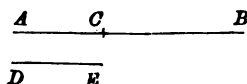
1.  $D$  is a point in the hypotenuse  $BC$  of a right-angled triangle  $ABC$ : prove that, if the rectangle  $BD, BC$  be equal to the square on  $AC$ , then the rectangle  $BC, DC$  is equal to the square on  $AB$ .
2. A point  $D$  is taken in the side  $BC$  of a triangle  $ABC$ : prove that, if the rectangles  $BD, BC$  and  $BC, DC$  be equal to the squares on  $AB, AC$  respectively, the angle  $BAC$  is a right angle.

## PROPOSITION 3.

*If a straight line be divided into any two parts, the rectangle contained by the whole line and one of the parts is equal to the sum of the square on that part and the rectangle contained by the two parts\*.*

Let the straight line  $AB$  be divided into any two parts at the point  $C$ :  
it is required to prove that the rectangle  $AB, AC$  is equal to the sum of the square on  $AC$  and the rectangle  $AC, CB$ .

CONSTRUCTION. Take  $DE$  a straight line equal to  $AC$ .



PROOF. The rectangle  $DE, AB$  is equal to the sum of the rectangles  $DE, AC$ , and  $DE, CB$ . (Prop. 1.)

Because  $DE$  is equal to  $AC$ ,  
the rectangle  $DE, AB$  is equal to the rectangle  $AC, AB$ ,  
the rectangle  $DE, AC$  to the square on  $AC$ ,  
and the rectangle  $DE, CB$  to the rectangle  $AC, CB$ ;  
therefore the rectangle  $AB, AC$  is equal to the sum of the square on  $AC$  and the rectangle  $AC, CB$ .

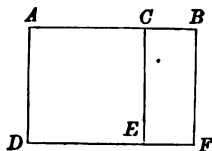
Wherefore, *if a straight line &c.*

\* The algebraical equivalent of this theorem is the equation

$$(a+b)a = a^2 + ab.$$

*Outline of Alternative Proof.*

On  $AC$  construct the square  $ADEC$ ,  
and draw  $BF$  parallel to  $AD$  to meet  $DE$  produced in  $F$ .



It may be proved that

- (1)  $AD$  is the rectangle  $AB, AC$ ,  
and (2)  $CF$  is the rectangle  $AC, CB$ ,  
and hence that the rectangle  $AB, AC$  is equal to the sum  
of the square on  $AC$  and the rectangle  $AC, CB$ .

**EXERCISES.**

1.  $AD$  is drawn perpendicular to the hypotenuse  $BC$  of a right-angled triangle  $ABC$ : prove that the rectangle  $BD, DC$  is equal to the square on  $AD$ .
2. In the triangle  $ABC$ ,  $AD$  is the perpendicular from  $A$  on  $BC$ : prove that, if the rectangle  $BD, DC$  be equal to the square on  $AD$ , the angle  $A$  is a right angle.
3. In the triangle  $ABC$ ,  $AD$  is the perpendicular from  $A$  on  $BC$ : prove that, if the rectangle  $BD, DC$  be equal to the square on  $AB$ , the angle  $A$  is a right angle.

## PROPOSITION 4.

*If a straight line be divided into any two parts, the square on the whole line is equal to the sum of the squares on the parts and twice the rectangle contained by the parts\*.*

Let the straight line  $AB$  be divided into any two parts at the point  $C$ :

it is required to prove that the square on  $AB$  is equal to the sum of the squares on  $AC$  and  $CB$ , and twice the rectangle contained by  $AC$ ,  $CB$ .

$\underline{A \qquad C \qquad B}$

PROOF. Because  $AB$  is divided into two parts at  $C$ , the square on  $AB$  is equal to the sum of the rectangles  $AB, AC$  and  $AB, CB$ , (Prop. 2.)

and the rectangle  $AB, AC$  is equal to the sum of the square on  $AC$  and the rectangle  $AC, CB$ , (Prop. 3.)

and the rectangle  $AB, CB$  is equal to the sum of the square on  $CB$  and the rectangle  $AC, CB$ . (Prop. 3.)

Therefore the square on  $AB$  is equal to the sum of the squares on  $AC$  and  $CB$ , and twice the rectangle  $AC, CB$ .

Wherefore, *if a straight line &c.*

\* The algebraical equivalent of this theorem is the equation

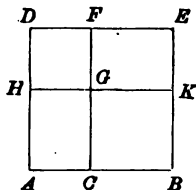
$$(a+b)^2 = a^2 + b^2 + 2ab.$$



*Outline of Alternative Proof.*

On  $AB$  construct the square  $ADEB$ :

draw  $CGF$  parallel to  $AD$ : make  $DH$  equal to  $AC$ ,  
and draw  $HGK$  parallel to  $AB$ .



It may be proved that

- (1)  $DG$  is equal to the square on  $AC$ ,
  - (2)  $GB$  is the square on  $CB$ ,
- and (3)  $HC, FK$  are each equal to the rectangle  $AC, CB$ ,  
and hence that the square on  $AB$  is equal to the sum of  
the squares on  $AC, CB$  and twice the rectangle  $AC, CB$ .

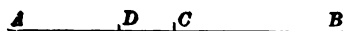
**EXERCISES.**

1. The square on a straight line is four times the square on half of the line.
2. If the sides  $BC, CA, AB$  of a right-angled triangle  $ABC$  be bisected in the points  $D, E, F$  respectively, twice the squares on  $AD, BE$  and  $CF$  are together equal to three times the square on the hypotenuse.
3. If, in an acute-angled triangle  $ABC$ ,  $AD$  be drawn perpendicular to  $BC$ , then the sum of the squares on  $AB, AC$  and twice the rectangle  $BD, DC$  is equal to the sum of the square on  $BC$  and twice the square on  $AD$ .
4. If  $BAC$  be an obtuse angle and  $BD, CE$  be drawn at right angles to  $CA, BA$  respectively, then the rectangle  $BA, AE$  is equal to the rectangle  $CA, AD$ .
5.  $ABCD$  is a square and  $E, F, G, H$  are points on the sides  $AB, BC, CD, DA$  respectively: prove that, if  $EFGH$  be a rectangle, it is either double of the rectangle  $AE, EB$ , or equal to the sum of the squares on  $AE, EB$ .

## PROPOSITION 5.

*If a straight line be divided into two equal parts and also into two unequal parts, the sum of the rectangle contained by the unequal parts and the square on the line between the points of section is equal to the square on half the line\*.*

Let the straight line  $AB$  be divided into two equal parts at the point  $C$ , and into two unequal parts at the point  $D$ : it is required to prove that the sum of the rectangle  $AD$ ,  $DB$  and the square on  $DC$ , is equal to the square on  $AC$ .



PROOF. Because  $DB$  is divided into two parts at  $C$ , the rectangle  $AD$ ,  $DB$  is equal to the sum of the rectangles  $AD$ ,  $DC$  and  $AD$ ,  $CB$ , (Prop. 1.) that is to the sum of the rectangles  $AD$ ,  $DC$ , and  $AD$ ,  $AC$ , since  $AC$  is equal to  $CB$ . (Hypothesis.)

And because  $AC$  is divided into two parts at  $D$ , the rectangle  $AD$ ,  $AC$  is equal to the sum of the square on  $AD$  and the rectangle  $AD$ ,  $DC$ . (Prop. 3.) Therefore the rectangle  $AD$ ,  $DB$  is equal to the sum of the square on  $AD$  and twice the rectangle  $AD$ ,  $DC$ .

Add to each of these equals the square on  $DC$ ; then the sum of the square on  $DC$  and the rectangle  $AD$ ,  $DB$  is equal to the sum of the squares on  $AD$ ,  $DC$  and twice the rectangle  $AD$ ,  $DC$ , which sum is equal to the square on  $AC$ . (Prop. 4.)

Therefore the rectangle  $AD$ ,  $DB$ , together with the square on  $DC$ , is equal to the square on  $AC$ .

Wherefore, if a straight line be

\* The algebraical equivalent of this theorem is the equation  
 $(a - b)(a + b) + b^2 = a^2$ .

The theorems of Propositions 5 and 6 may both be included in one enunciation, thus, *The difference of the squares on two given straight lines is equal to the rectangle contained by the sum and the difference of the lines.*

The straight lines in both propositions are  $AD$ ,  $BD$ : the only difference being that in Prop. 5.  $BD$  is the greater and in Prop. 6.  $AD$  is the greater.

For an outline of an alternative proof of Propositions 5 and 6, see page 147.

#### EXERCISES.

1. A straight line is divided into two parts; shew that, if twice the rectangle of the parts be equal to the sum of the squares on the parts, the straight line is bisected.
2. Divide a given straight line into two parts such that the rectangle contained by them shall be the greatest possible.
3. Divide a given straight line into two parts such that the sum of the squares on the two parts may be the least possible.
4. Divide a given straight line into three parts so that the sum of the squares on them may be the least possible.
5.  $ABC$  is an equilateral triangle and  $D$  is any point in the side  $BC$ . Prove that the square on  $BC$  is equal to the rectangle contained by  $BD$ ,  $DC$ , together with the square on  $AD$ .
6. A point  $D$  is taken on the hypotenuse  $BC$  of a right-angled triangle  $ABC$ ; prove that, if the rectangle  $BD$ ,  $DC$  be equal to the square on  $AD$ ,  $D$  is either the middle point of  $BC$  or the foot of the perpendicular from  $A$  on  $BC$ .

## PROPOSITION 6.

*If a straight line be bisected, and produced to any point, the sum of the rectangle contained by the whole line thus produced and the part of it produced, and the square on half the line bisected, is equal to the square on the straight line which is made up of the half and the part produced\*.*

Let the straight line  $AB$  be bisected at the point  $C$ , and produced to the point  $D$ :  
it is required to prove that the sum of the rectangle  $AD$ ,  $BD$ , and the square on  $CB$  is equal to the square on  $CD$ .



PROOF. Because  $AD$  is divided into two parts at  $B$ , the rectangle  $AD$ ,  $BD$  is equal to the sum of the rectangle  $AB$ ,  $BD$  and the square on  $BD$ . (Prop. 3.)

Because  $AB$  is bisected in  $C$ , the rectangle  $AB$ ,  $BD$  is double of the rectangle  $CB$ ,  $BD$ ; (Prop. 1.)  
therefore the rectangle  $AD$ ,  $BD$  is equal to the sum of the square on  $BD$  and twice the rectangle  $CB$ ,  $BD$ .

Add to each of these equals the square on  $CB$ ;  
then the sum of the square on  $CB$  and the rectangle  $AD$ ,  $BD$  is equal to the sum of the squares on  $CB$ ,  $BD$  and twice the rectangle  $CB$ ,  $BD$ .

And the sum of the squares on  $CB$ ,  $BD$  and twice the rectangle  $CB$ ,  $BD$  is equal to the square on  $CD$ ; (Prop. 4.)

therefore the sum of the rectangle  $AD$ ,  $BD$  and the square on  $CB$  is equal to the square on  $CD$ .

Wherefore, *if a straight line &c.*

\* The algebraical equivalent of this theorem is the equation

$$(a + b)(b - a) + a^2 = b^2,$$

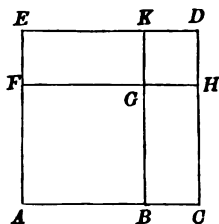
or

$$(2a + b)b + a^2 = (a + b)^2.$$

*Outline of Alternative Proof of Propositions 5 and 6.*

Let  $A, B, C$  be any three points in a straight line.

Draw  $AFE, BGK, CHD$  at right angles to  $ABC$ .



Take  $AF$  equal to  $AB$  and  $FE$  to  $BC$ , and draw  $EKD, FGH$  parallel to  $ABC$ .

It may be proved that

- (1)  $FB$  is the square on  $AB$ ,
  - (2)  $KH$  is equal to the square on  $BC$ ,
  - (3).  $EB$  is the rectangle  $AB, AC$ ,
- and (4)  $EH$  is the rectangle  $AC, BC$ ,
- and hence that the difference of the squares on  $AB, BC$ , which is equal to the difference of the rectangles  $EB, EH$ , is equal to the rectangle contained by the sum and the difference of  $AB$  and  $BC$ .

#### EXERCISES.

1. A straight line is divided into two equal and also into two unequal parts; prove that the difference of the squares on the two unequal parts is equal to twice the rectangle contained by the whole line and the part between the points of section.

2. The straight line  $AB$  is bisected in  $C$  and produced to  $D$ ,  $CE$  is drawn perpendicular to  $AB$  and equal to  $BD$ , and a point  $F$  is taken in  $BD$  so that  $EF$  is equal to  $CD$ ; prove that the rectangle  $DF, DA$  together with the rectangle  $DF, FB$  is equal to the square on  $BD$ .

## PROPOSITION 7.

*If a straight line be divided into any two parts, the sum of the squares on the whole line and on one of the parts is equal to the sum of twice the rectangle contained by the whole line and that part and the square on the other part\*.*

Let the straight line  $AB$  be divided into any two parts at the point  $C$ :

it is required to prove that the sum of the squares on  $AB$ ,  $CB$  is equal to the sum of twice the rectangle  $AB$ ,  $CB$ , and the square on  $AC$ .



PROOF. Because  $AB$  is divided into two points at  $C$ , the square on  $AB$  is equal to the sum of the squares on  $AC$ ,  $CB$ , and twice the rectangle  $AC$ ,  $CB$ . (Prop. 4.)

Add to each of these equals the square on  $CB$ ; then the sum of the squares on  $AB$ ,  $CB$  is equal to the sum of the square on  $AC$ , twice the square on  $CB$ , and twice the rectangle  $AC$ ,  $CB$ .

But the sum of the square on  $CB$  and the rectangle  $AC$ ,  $CB$  is equal to the rectangle  $AB$ ,  $CB$ . (Prop. 3.)

Therefore the sum of the squares on  $AB$ ,  $CB$ , is equal to the sum of twice the rectangle  $AB$ ,  $CB$ , and the square on  $AC$ .

Wherefore, *if a straight line &c.*

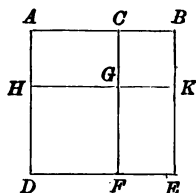
COROLLARY. *If a straight line be divided into any two parts, the square on one of the parts is less than the sum of the squares on the whole line and the other part by twice the rectangle contained by the whole line and the second part.*

\* The algebraical equivalent of this theorem is the equation  
 $(a+b)^2 + b^2 = 2(a+b)b + a^2$ .

*Outline of Alternative Proof.*

On  $AB$  construct the square  $ADEB$   
and draw  $CGF$  parallel to  $AD$ .

Take  $BK$  equal to  $BC$ ,  
and draw  $KGH$  parallel to  $BCA$ .



It may be proved that

- (1)  $AK, CE$  are each equal to the rectangle  $AB, CB$ ,
  - (2)  $HF$  is equal to the square on  $AC$ ,
- and (3)  $CK$  is the square on  $CB$ ,
- and hence that the sum of the squares on  $AB, CB$  is equal to the sum of twice the rectangle  $AB, CB$  and the square on  $AC$ .

## EXERCISES.

1.  $ACDB$  is a straight line, and  $D$  bisects  $CB$ : prove that the square on  $AC$  is less than the sum of the squares on  $AD, DB$  by twice the rectangle  $AD, DB$ .
2. If  $BAC$  be an acute angle and  $BD, CE$  be drawn perpendicular to  $CA, AB$  respectively, then the rectangle  $BA, AE$  is equal to the rectangle  $CA, AD$ .
3. Shew how to divide a given straight line into two parts such that the difference of the squares described on them may be equal to a given rectangle. Is a solution always possible?

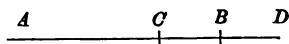
## PROPOSITION 8.

*If a straight line be divided into any two parts, the sum of the square on one part and four times the rectangle contained by the whole line and the other part, is equal to the square on the straight line which is made up of the whole and the second part\*.*

Let the straight line  $AB$  be divided into any two parts at the point  $C$ :

it is required to prove that the sum of four times the rectangle  $AB, CB$ , and the square on  $AC$  is equal to the square on the straight line made up of  $AB$  and  $CB$  together.

CONSTRUCTION. Produce  $AB$  to  $D$ ,  
and make  $BD$  equal to  $CB$ . (I. Prop. 3.)



PROOF. Because  $AD$  is divided into two parts at  $B$ , the sum of the squares on  $AB, BD$  and twice the rectangle  $AB, BD$ , is equal to the square on  $AD$ : (Prop. 4.)

and because  $AB$  is divided into two parts at  $C$ , the sum of the square on  $AC$  and twice the rectangle  $AB, CB$ , is equal to the sum of the squares on  $AB, CB$ . (Prop. 7.)

Add these equals together;  
then the sum of the squares on  $AB, BD, AC$  and four times the rectangle  $AB, CB$ ,  
is equal to the sum of the squares on  $AB, CB, AD$ .

Take away from these equals, the equals the sum of the squares on  $AB, BD$  and the sum of the squares on  $AB, CB$ ;

then the sum of the square on  $AC$  and four times the rectangle  $AB, CB$  is equal to the square on  $AD$ .

Wherefore, *if a straight line &c.*

\* The algebraical equivalent of this theorem is the equation

$$(a-b)^2 + 4ab = (a+b)^2,$$

or

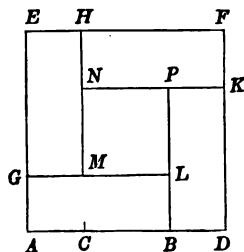
$$a^2 + 4(a+b)b = (a+2b)^2.$$



*Outline of Alternative Proof.*

Produce  $AB$  to  $D$  and make  $BD$  equal to  $AC$ .

On  $AD$  construct the square  $AEFD$ .



Take  $AG$ ,  $EH$ ,  $FK$  each equal to  $AC$ ,  
and draw  $BLP$ ,  $MNH$  parallel to  $AE$  and  $GML$ ,  $NPK$   
parallel to  $AD$ .

It may be proved that

- (1)  $AL$ ,  $BK$ ,  $FN$ ,  $EM$  are each equal to the rectangle  $AB$ ,  $AC$ ,  
and (2)  $MP$  is equal to the square on  $CB$ ,  
and hence that the sum of the square on  $AC$  and four times  
the rectangle  $AB$ ,  $CB$  is equal to the square on  $AD$ .

## EXERCISES.

1. Prove that the square on a straight line is nine times the square on one third of the line.

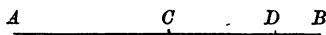
2. If a straight line be bisected and produced to any point, the square on the whole line thus produced is equal to the square on the part produced together with twice the rectangle contained by the line and the line made up of the half and the part produced.

## PROPOSITION 9.

*If a straight line be divided into two equal, and also into two unequal parts, the sum of the squares on the two unequal parts is double of the sum of the squares on half the line and on the line between the points of section\*.*

Let the straight line  $AB$  be divided into two equal parts at the point  $C$ , and into two unequal parts at the point  $D$ :

it is required to prove that the sum of the squares on  $AD$ ,  $DB$  is double of the sum of the squares on  $AC$ ,  $CD$ .



**PROOF.** Because  $AD$  is divided at  $C$ , the square on  $AD$  is equal to the sum of the squares on  $AC$ ,  $CD$  and twice the rectangle  $AC$ ,  $CD$ . (Prop. 4.)

And because  $CB$  is divided at  $D$ , the sum of the square on  $DB$  and twice the rectangle  $CB$ ,  $CD$  is equal to the sum of the squares on  $CB$ ,  $CD$ . (Prop. 7.)

Add these pairs of equals;  
then the sum of the squares on  $AD$ ,  $DB$  and twice the rectangle  $CB$ ,  $CD$  is equal to the sum of the squares on  $AC$ ,  $CD$ ,  $CB$ ,  $CD$  and twice the rectangle  $AC$ ,  $CD$ .  
Take away from these equals twice the rectangle  $CB$ ,  $CD$ , and twice the rectangle  $AC$ ,  $CD$ , which are equal;  
then the sum of the squares on  $AD$ ,  $DB$  is equal to the sum of the squares on  $AC$ ,  $CD$ ,  $CB$ ,  $CD$ ,  
that is, is equal to twice the sum of the squares on  $AC$ ,  $CD$ .  
Wherefore, *if a straight line &c.*

\* The algebraical equivalent of this theorem is the equation

$$(a+b)^2 + (a-b)^2 = 2(a^2 + b^2).$$

The theorems of Propositions 9 and 10 may both be included in one enunciation: thus, *The sum of the squares on the sum and on the difference of two given straight lines is equal to twice the sum of the squares on the lines.*

The straight lines in both propositions are  $AC$ ,  $CD$ : the only difference being that in Prop. 9.  $AC$  is the greater, and in Prop. 10.  $CD$  is the greater.

For an outline of an alternative proof of Propositions 9 and 10, see page 155.

### EXERCISES.

1. A straight line is divided into two parts, such that the diagonal of the square on one of these parts is equal to the whole line. If a square be constructed whose side is the difference between the aforesaid part and half the given line, its diagonal is equal to the other of the two parts into which the line is divided.

2. Deduce a proof of II. 9 from the result of II. 5.

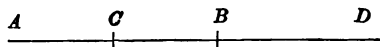
3. If a straight line be divided into two equal and also into two unequal parts, the squares on the two unequal parts are equal to twice the rectangle contained by the two unequal parts together with four times the square on the line between the points of section.

## PROPOSITION 10.

*If a straight line be bisected and produced to any point, the sum of the squares on the whole line thus produced and on the part produced is double of the sum of the squares on half the line and on the line made up of the half and the part produced\*.*

Let the straight line  $AB$  be bisected at  $C$ , and produced to  $D$ :

it is required to prove that the sum of the squares on  $AD$ ,  $BD$  is double of the sum of the squares on  $AC$ ,  $CD$ .



PROOF. Because  $AD$  is divided at  $C$ ,  
the square on  $AD$  is equal to the sum of the squares on  
 $AC$ ,  $CD$  and twice the rectangle  $AC$ ,  $CD$ ; (Prop. 4.)  
and because  $CD$  is divided at  $B$ ,  
the sum of the square on  $BD$  and twice the rectangle  $CB$ ,  $CD$   
is equal to the sum of the squares on  $CB$ ,  $CD$ .

(Prop. 7).

Add these equals;

then the sum of the squares on  $AD$ ,  $BD$  and twice the  
rectangle  $CB$ ,  $CD$  is equal to the sum of the squares on  
 $AC$ ,  $CD$ ,  $CB$ ,  $CD$  and twice the rectangle  $AC$ ,  $CD$ .

Take away from these equals twice the rectangle  $CB$ ,  $CD$ ,  
and twice the rectangle  $AC$ ,  $CD$ , which are equal;  
then the sum of the squares on  $AD$ ,  $BD$  is equal to the  
sum of the squares on  $AC$ ,  $CD$ ,  $CB$ ,  $CD$ ,  
that is, is equal to twice the sum of the squares on  $AC$ ,  $CD$ .

Wherefore, *if a straight line &c.*

\* The algebraical equivalent of this theorem is the equation

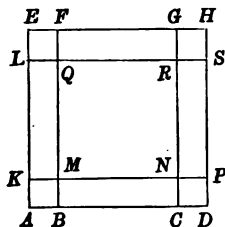
$$(a+b)^2 + (b-a)^2 = 2(a^2 + b^2).$$

*Outline of Alternative Proof of Propositions 9 and 10.*

In a straight line  $ABCD$ , take  $AB$  equal to  $CD$ .  
Through  $A, B, C, D$  draw  $AE, BF, CG, DH$  at right angles to  $ABCD$ .

Take  $AK$  equal to  $AB$ ,  $KL$  to  $BC$  and  $LE$  to  $AB$ .

Draw  $EFGH, LQRS, KMNP$  parallel to  $ABCD$ .



It may be proved that

- (1)  $EQ, ND$  are each equal to the square on  $AB$ ,
  - (2)  $LC, FP$  are each equal to the square on  $AC$ ,
  - (3)  $QN$  is equal to the square on  $BC$ ,
  - (4)  $ED$  is the square on  $AD$ ,
- and (5) the sum of  $ED$  and  $QN$  is equal to the sum of  $EQ, ND, LC$ , and  $FP$ ,  
and hence that the sum of the squares on  $AD, BC$  (which are the sum and the difference of  $AC$  and  $AB$ ) is equal to twice the sum of the squares on  $AC, AB$ .

#### EXERCISES.

1. In  $AB$  the diameter of a circle take two points  $C$  and  $D$  equally distant from the centre, and from any point  $E$  in the circumference draw  $EC, ED$ : shew that the squares on  $EC$  and  $ED$  are together equal to the squares on  $AC$  and  $AD$ .

2. If in  $BC$  the base of a triangle a point  $D$  be taken such that the squares on  $AB$  and  $BD$  are together equal to the squares on  $AC$  and  $CD$ , then the middle point of  $AD$  will be equally distant from  $B$  and  $C$ .

3. A square  $BDEC$  is described on the hypotenuse  $BC$  of a right-angled triangle  $ABC$ : shew that the squares on  $DA$  and  $AC$  are together equal to the squares on  $EA$  and  $AB$ .

4.  $AB$  is divided into any two parts in  $C$ , and  $AC, BC$  are bisected in  $D, E$ : prove that the square on  $AE$  and three times the square on  $BE$  are equal to the square on  $BD$  and three times the square on  $AD$ .

## PROPOSITION 11.

*To divide a given straight line into two parts, so that the rectangle contained by the whole and one part may be equal to the square on the other part\*.*

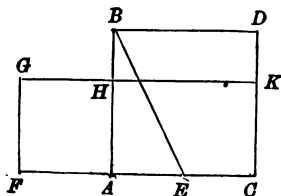
Let  $AB$  be the given straight line:  
it is required to divide it into two parts in a point  $H$ , so that the rectangle contained by the whole line  $AB$  and a part  $HB$  may be equal to the square on the other part  $AH$ .

CONSTRUCTION. On  $AB$  construct the square  $ABDC$ ;  
(I. Prop. 46.)  
bisect  $AC$  at  $E$ ;  
(I. Prop. 10.)

draw  $BE$ ; produce  $CA$  to  $F$ ,  
and make  $EF$  equal to  $EB$ ; (I. Prop. 3.)

and on  $AF$  construct the square  $AFGH$ : (I. Prop. 46.)  
then  $AB$  is divided at  $H$  so that the rectangle  $AB, HB$  is  
equal to the square on  $AH$ .

Produce  $GH$  to meet  $CD$  at  $K$ .



PROOF. Because  $AC$  is bisected at  $E$ , and produced to  $F$ ,  
the sum of the rectangle  $FC, FA$ , and the square on  $AE$  is  
equal to the square on  $FE$ . (Prop. 6.)

But  $FE$  is equal to  $EB$ . (Constr.)

Therefore the sum of the rectangle  $FC, FA$ , and the square  
on  $AE$  is equal to the square on  $EB$ .

But, because the angle  $EAB$  is a right angle,  
the square on  $EB$  is equal to the sum of the squares on  
 $AE, AB$ . (I. Prop. 47.)

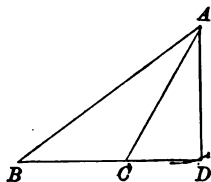
\* The algebraical equivalent of this problem is to find the smaller  
root of the quadratic equation  $ax = (a - x)^2$ , or the positive root of the  
quadratic equation  $a(a - x) = x^2$ .



## PROPOSITION 12.

*In an obtuse-angled triangle, if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square on the side opposite the obtuse angle is greater than the sum of the squares on the other sides, by twice the rectangle contained by the side on which, when produced, the perpendicular falls, and the part of the produced side intercepted between the perpendicular and the obtuse angle.*

Let  $ABC$  be an obtuse-angled triangle, and let the angle  $ACB$  be the obtuse angle; from the point  $A$  let  $AD$  be drawn perpendicular to  $BC$  produced: it is required to prove that the square on  $AB$  is greater than the sum of the squares on  $AC$ ,  $CB$ , by twice the rectangle  $BC$ ,  $CD$ .



**PROOF.** Because  $BD$  is divided into two parts at  $C$ , the square on  $BD$  is equal to the sum of the squares on  $BC$ ,  $CD$ , and twice the rectangle  $BC$ ,  $CD$ . (Prop. 4.)

To each of these equals add the square on  $DA$ ; then the sum of the squares on  $BD$ ,  $DA$  is equal to the sum of the squares on  $BC$ ,  $CD$ ,  $DA$ , and twice the rectangle  $BC$ ,  $CD$ .

But, because the angle at  $D$  is a right angle, the square on  $BA$  is equal to the sum of the squares on  $BD$ ,  $DA$ , and the square on  $CA$  is equal to the sum of the squares on  $CD$ ,  $DA$ . (I. Prop. 47.)

Therefore the square on  $BA$  is equal to the sum of the squares on  $BC$ ,  $CA$ , and twice the rectangle  $BC$ ,  $CD$ ; that is, the square on  $BA$  is greater than the sum of the squares on  $BC$ ,  $CA$  by twice the rectangle  $BC$ ,  $CD$ .

Wherefore *in an obtuse-angled triangle &c.*

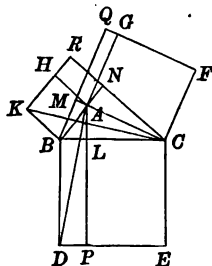


*Outline of Alternative Proof.*

On the sides  $BC$ ,  $CA$ ,  $AB$  of an obtuse-angled triangle  $ABC$ , in which the angle  $BAC$  is obtuse, construct the squares  $BDEC$ ,  $CFGA$ ,  $AHKB$ .

Draw  $AL$ ,  $BM$ ,  $CN$  perpendicular to  $BC$ ,  $CA$ ,  $AB$  and produce them to meet the opposite sides (produced if necessary) of the squares in  $P$ ,  $Q$ ,  $R$ .

Draw  $AD$ ,  $CK$ .



It may be proved that

- (1) the triangle  $ABD$  is equal to the triangle  $KBC$  in all respects,  
 and (2) the rectangle  $BP$  is equal to the rectangle  $BR$ ,  
 and similarly that  $CQ$  is equal to  $CP$ , and  $AR$  to  $AQ$ ,  
 and hence that the square on  $BC$  is greater than the  
 sum of the squares on  $CA$ ,  $AB$  by twice the rectangle  
 $AR$  or  $AQ$ .

## EXERCISES.

1. The sides of a triangle are 10, 12, 15 inches: prove that it is acute-angled.
2. On the side  $BC$  of any triangle  $ABC$ , and on the side of  $BC$  remote from  $A$ , a square  $BDEC$  is constructed. Prove that the difference of the squares on  $AB$  and  $AC$  is equal to the difference of the squares on  $AD$  and  $AE$ .
3.  $C$  is the obtuse angle of a triangle  $ABC$ , and  $D$ ,  $E$  the feet of the perpendiculars drawn from  $A$ ,  $B$  respectively to the opposite sides produced: prove that the square on  $AB$  is equal to the sum of the rectangles contained by  $BC$ ,  $BD$  and  $AC$ ,  $AE$ .
4.  $ABC$  is a triangle having the sides  $AB$  and  $AC$  equal;  $AB$  is produced beyond the base to  $D$  so that  $BD$  is equal to  $AB$ ; shew that the square on  $CD$  is equal to the square on  $AB$ , together with twice the square on  $BC$ .

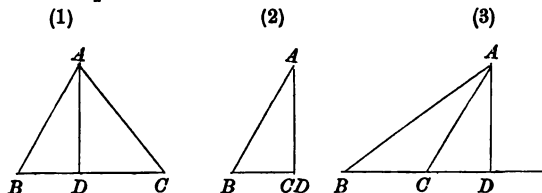
## PROPOSITION 13.

*In any triangle, the square on a side subtending an acute angle, is less than the sum of the squares on the other sides, by twice the rectangle contained by either of these sides, and the part of the side intercepted between the perpendicular let fall on it from the opposite angle, and the acute angle.*

Let  $ABC$  be a triangle, and let the angle  $ABC$  be an acute angle; let  $AD$  be drawn perpendicular to  $BC$  and meet it (produced if necessary) in  $D$ :

it is required to prove that the square on  $AC$  is less than the sum of the squares on  $AB$ ,  $BC$  by twice the rectangle  $BC$ ,  $BD$ .

Either (1)  $D$  lies in  $BC$ , or (2)  $D$  coincides with  $C$ , or (3)  $D$  lies in  $BC$  produced.



PROOF. Because fig. (1)  $BC$  is divided in  $D$ , fig. (2)  $D$  is the same point as  $C$ , or fig. (3)  $BD$  divided in  $C$ , the sum of the squares on  $BC$ ,  $BD$  is equal to the sum of the square on  $CD$ , and twice the rectangle  $BC$ ,  $BD$ .

(I. Prop. 47 and II. Prop. 7.)

To each of these equals add the square on  $DA$ ; then the sum of the squares on  $BC$ ,  $BD$ ,  $DA$  is equal to the sum of the squares on  $CD$ ,  $DA$  and twice the rectangle  $BC$ ,  $BD$ .

But because the angle  $BDA$  is a right angle, the square on  $AB$  is equal to the sum of the squares on  $BD$ ,  $DA$ , and the square on  $AC$  is equal to the sum of the squares on  $CD$ ,  $DA$ .

(I. Prop. 47.)

Therefore the sum of the squares on  $BC$ ,  $AB$  is equal to the sum of the square on  $AC$  and twice the rectangle  $BC$ ,  $BD$ : that is, the square on  $AC$  is less than the sum of the squares on  $AB$ ,  $BC$  by twice the rectangle  $BC$ ,  $BD$ .

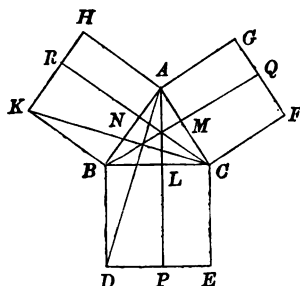
Wherefore, in any triangle &c.

*Outline of Alternative Proof.*

On the sides  $BC$ ,  $CA$ ,  $AB$  of an acute-angled triangle  $ABC$  construct the squares  $BDEC$ ,  $CFGA$ ,  $AHKB$ .

Draw  $AL$ ,  $BM$ ,  $CN$  perpendicular to  $BC$ ,  $CA$ ,  $AB$ , and produce them to meet the opposite sides of the squares in  $P$ ,  $Q$ ,  $R$ .

Draw  $AD$ ,  $CK$ .



It may be proved that

- (1) the triangle  $ABD$  is equal to the triangle  $KBC$  in all respects,
- and (2) the rectangle  $BP$  is equal to the rectangle  $BR$ , and similarly that  $CQ$  is equal to  $CP$ , and  $AR$  to  $AQ$ , and hence that the square on  $BC$  is less than the sum of the squares on  $CA$ ,  $AB$  by twice the rectangle  $AQ$  or  $AR$ .

**EXERCISES.**

1. In any triangle the sum of the squares on the sides is equal to twice the square on half the base together with twice the square on the straight line drawn from the vertex to the middle point of the base.
2. The base of a triangle is given: find the locus of the vertex, when the sum of the squares on the two sides is given.
3. The sum of the squares on the sides of a parallelogram is equal to the sum of the squares on the diagonals.
4. In any quadrilateral the squares on the diagonals are together equal to twice the sum of the squares on the straight lines joining the middle points of opposite sides.
5. The squares on the sides of a quadrilateral are together greater than the squares on its diagonals by four times the square on the straight line joining the middle points of its diagonals.

## PROPOSITION 14.

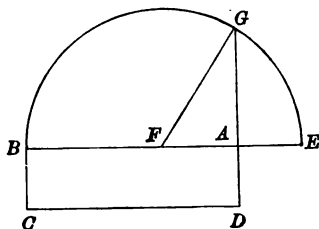
*To find the side of a square equal to a given rectangle.\**

Let  $ABCD$  be the given rectangle :  
it is required to find the side of a square equal to  $ABCD$ .

CONSTRUCTION. If two adjacent sides  $BA$ ,  $AD$  be equal,  
the rectangle is a square, and  $BA$  or  $AD$  is the line required.

But if they be not equal,  
produce one of them  $BA$  to  $E$ ,  
and make  $AE$  equal to  $AD$ ; (I. Prop. 3.)  
bisect  $BE$  at  $F$ ; (I. Prop. 10.)  
and with  $F$  as centre and  $FB$  as radius,  
describe the circle  $BGE$ ,  
and produce  $DA$  to meet the circle in  $G$ ;  
then  $AG$  is the line required.

Draw  $FG$ .



PROOF. Because  $BE$  is divided into two equal parts  
at  $F$ , and into two unequal parts at  $A$ ,  
the sum of the rectangle  $BA$ ,  $AE$  and the square on  $FA$  is  
equal to the square on  $FE$ . (Prop. 5.)

But  $FE$  is equal to  $FG$ .  
Therefore the sum of the rectangle  $BA$ ,  $AE$  and the square  
on  $FA$  is equal to the square on  $FG$ .

\* The algebraical equivalent of this problem is to find the positive  
root of the quadratic equation  $x^2 = ab$ .

But because the angles at  $A$  are right angles, the square on  $FG$  is equal to the sum of the squares on  $FA$ ,  $AG$ ;

(I. Prop. 47.)

therefore the sum of the rectangle  $BA$ ,  $AE$  and the square on  $FA$ , is equal to the sum of the squares on  $FA$ ,  $AG$ .

Take away from each of these equals the square on  $FA$ ; then the rectangle  $BA$ ,  $AE$  is equal to the square on  $AG$ .

But  $ABCD$  is the rectangle contained by  $BA$ ,  $AE$ , since it is contained by  $BA$ ,  $AD$ , and  $AE$  is equal to  $AD$ .

Therefore  $ABCD$  is equal to the square on  $AG$ .

Wherefore *the straight line  $AG$  has been found, which is the side of a square equal to the given rectangle  $ABCD$ .*

COROLLARY 1. *To find the side of a square equal to a given rectilineal figure.*

Construct a rectangle, i.e. a parallelogram, which has an angle equal to a right angle, equal to the given figure, (I. Prop. 45), and then use Proposition 14.

COROLLARY 2. *The square on the perpendicular from any point of a circle on any diameter is equal to the rectangle contained by the parts of that diameter intercepted between its extremities and the perpendicular.*

### EXERCISES.

1. Construct a rectangle equal to a given square, and having (1) the sum (2) the difference of two of its adjacent sides equal to a given straight line.

2. The largest rectangle, the sum of whose sides is given, is a square.

3. Construct a rectangle equal to a given square such that the difference of two adjacent sides shall be equal to a given straight line.

4. Construct a right-angled triangle equal to a given rectilineal figure and such that one of the sides containing the right angle is double of the other.

5. Produce a given straight line  $AB$  both ways to  $C$  and  $D$  so that the rectangles  $CA$ ,  $AD$  and  $CB$ ,  $BD$  may be equal respectively to two given squares.

## MISCELLANEOUS EXERCISES.

1. In a triangle whose vertical angle is a right angle a straight line is drawn from the vertex perpendicular to the base: shew that the square on either of the sides adjacent to the right angle is equal to the rectangle contained by the base and the segment of it adjacent to that side.

2. If  $ABC$  be a triangle whose angle  $A$  is a right angle, and  $BE$ ,  $CF$  be drawn bisecting the opposite sides, four times the sum of the squares on  $BE$  and  $CF$  is equal to five times the square on  $BC$ .

3. The hypotenuse  $AB$  of a right-angled triangle  $ABC$  is trisected in the points  $D$ ,  $E$ ; shew that, if  $CD$ ,  $CE$  be joined, the sum of the squares on the three sides of the triangle  $CDE$  is equal to two thirds of the square on  $AB$ .

4.  $ABCD$  is a rectangle, and points  $E$ ,  $F$  are taken in  $BC$ ,  $CD$  respectively. Prove that twice the area of the triangle  $AEF$  together with the rectangle  $BE$ ,  $DF$  is equal to the rectangle  $AB$ ,  $BC$ .

5. On the outside of the hypotenuse  $BC$ , and the sides  $CA$ ,  $AB$  of a right-angled triangle  $ABC$ , squares  $BDEC$ ,  $AF$ , and  $AG$  are described: shew that the squares on  $DG$  and  $EF$  are together equal to five times the square on  $BC$ .

6. On the outside of the hypotenuse  $BC$  of a right-angled triangle  $ABC$  and on the sides  $CA$ ,  $AB$  squares  $BDEC$ ,  $AF$ , and  $AG$  are constructed: prove that the difference of the squares on  $DG$  and  $EF$  is three times of the difference of the squares on  $AB$  and  $AC$ .

7. In the hypotenuse  $AB$  of a right-angled triangle  $ACB$ , points  $D$  and  $E$  are taken such that  $AD$  is equal to  $AC$  and  $BE$  to  $BC$ ; prove that the square on  $DE$  is equal to twice the rectangle  $BD$ ,  $AE$ .

8. One diagonal  $AC$  of a rhombus  $ABCD$  is divided into any two parts at the point  $P$ ; shew that the rectangle  $AP$ ,  $PC$  is equal to the difference between the squares on  $AB$  and  $PB$ .

9. In a given straight line find a point such that the difference of the squares upon the straight lines joining it to two given points may be equal to a given rectangle. In what cases is this problem impossible?

10. If a straight line be drawn through one of the angles of an equilateral triangle to meet the opposite side produced, so that the rectangle contained by the whole straight line thus produced and the part of it produced is equal to the square on the side of the triangle, the square on the straight line so drawn will be double the square on a side of the triangle.

11. The square on any straight line drawn from the vertex of an isosceles triangle to any point in the base is less than the square on a side of the triangle by the rectangle contained by the segments of the base.

12. In the figure of II. 11,  $BE$  and  $CH$  meet at  $O$ ; shew that  $AO$  is at right angles to  $CH$ .

13. Divide a given straight line so that the square on one part is equal to the rectangle contained by the other part and another given straight line.

14. Divide a given straight line so that the rectangle contained by the parts shall be equal to the rectangle contained by the whole line and the difference of the parts.

15. How many triangles can be formed by choosing three lines out of six whose lengths are 3, 4, 5, 11, 12 and 13 inches? How many of these triangles are (1) obtuse-angled, (2) right-angled, (3) acute-angled?

16. Prove that the sum of the squares on the straight lines drawn from any point to the middle points of the sides of a triangle is less than the sum of the squares on the straight lines drawn from the same point to the angular points of the triangle by one quarter the sum of the squares on the sides of the triangle.

17.  $ABDC$  is a parallelogram whose diagonals intersect in  $O$ .  $OL$ ,  $OM$  are drawn at right angles to  $AB$ ,  $AC$  meeting them in  $L$ ,  $M$  respectively; prove that the sum of the rectangles  $AB$ ,  $AL$  and  $AC$ ,  $AM$  is double the square on  $AO$ .

18. In a triangle  $ABC$  the angles  $B$  and  $C$  are acute: if  $E$ ,  $F$  be the points where perpendiculars from the opposite angles meet the sides  $AC$ ,  $AB$ , the square on  $BC$  is equal to the rectangle  $AB$ ,  $BF$ , together with the rectangle  $AC$ ,  $CE$ . What change is made in the theorem, if  $B$  be an obtuse angle?

19. Prove that the locus of a point, whose distance from one of two fixed points is double that from the other, is a circle.

20. At  $B$  in  $AB$  two equal straight lines  $BC$ ,  $BD$  are drawn making equal angles with  $AB$  and  $AB$  produced. Shew that the difference of the squares on  $AC$  and  $AD$  is equal to twice the rectangle  $AB$ ,  $CD$ .

21. If the squares on the sides of a quadrilateral be equal to the squares on the diagonals, it must be a parallelogram.

22.  $ABC$  is a triangle having the angle  $C$  greater than the angle  $B$ , and  $D$  a point in the base  $BC$ , such that the sum of the squares on  $AB$ ,  $AC$  is twice the sum of the squares on  $AD$ ,  $DB$ . Shew that either  $D$  is the middle point of the base, or that its distance from the foot of the perpendicular from  $A$  on  $BC$  is one half of  $BC$ .

23. If a figure be composed of a rhombus and the square described outwards on one of its sides, the side of the equivalent square is equal to half the sum of the diagonals of the rhombus.

24.  $ABC$  is a triangle in which  $C$  is a right angle, and  $DE$  is drawn from a point  $D$  in  $AC$  perpendicular to  $AB$ ; shew that the rectangle  $AB$ ,  $AE$  is equal to the rectangle  $AC$ ,  $AD$ .

25.  $ABC$  is an acute-angled triangle; perpendiculars  $APD$ ,  $BPE$  are drawn on  $BC$ ,  $CA$  from the opposite angles. Prove that the rectangle  $BD$ ,  $DC$  is equal to the rectangle  $AD$ ,  $PD$ .

26.  $A$ ,  $B$ ,  $C$ ,  $D$  are the angular points of a parallelogram, taken in order. If there be a point  $P$  such that the sum of the squares on  $PA$  and  $PC$  be equal to the sum of those on  $PB$  and  $PD$ ,  $ABCD$  is a rectangle.

27.  $A$ ,  $B$ ,  $C$ ,  $D$  are fixed points, and  $P$  is a point such that the sum of the squares on  $PA$ ,  $PB$ ,  $PC$ ,  $PD$  is constant; prove that  $P$  lies on a circle, the centre of which is at the point where the straight line joining the middle points of  $AB$ ,  $CD$  cuts the straight line joining the middle points of  $AD$ ,  $BC$ .

28.  $A$ ,  $B$ ,  $C$ ,  $D$  are fixed points, and  $P$  is a point such that the sum of the squares on  $PA$ ,  $PB$  is equal to the sum of the squares on  $PC$ ,  $PD$ . Prove that the locus of  $P$  is a straight line at right angles to the line joining the middle points of  $AB$ ,  $CD$ , and passes through the intersection of the lines drawn perpendicular to either of the pairs of lines  $AC$ ,  $BD$  or  $AD$ ,  $BC$  at their middle points.

29. In the plane of a triangle  $ABC$  find a point  $P$  such that the sum of the squares on  $AP$ ,  $BP$  and  $CP$  may be a minimum.

30. Produce a given straight line so that the rectangle contained by the whole line thus produced and the produced part may be equal to the square on another given straight line.



## BOOK III.

### DEFINITIONS.

**DEFINITION 1.** *Any part of a circle is called an arc.*

The line which has been defined (1. Def. 22) as a circle is often spoken of as **the circumference** of the circle.

The reason of this is that a circle is defined in many books as the part of the plane contained by the line, which is then called the circumference.

*Half of a circle is called a semicircle.*

It will be proved hereafter that a diameter bisects a circle, i.e. divides it into two equal arcs. (See page 175.)

**DEFINITION 2.** *A straight line joining two points on a circle is called a chord of the circle.*

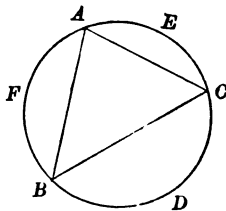
*The straight line joining the extremities of an arc is called the chord of the arc.*

The figure formed of an arc and the chord of the arc is called **a segment of the circle**.

In the diagram the straight lines  $AB$ ,  $BC$ ,  $CA$  are **chords** of the circle  $ABC$ ;  $AFB$ ,  $BDC$ ,  $CEA$  are **arcs**.

The straight line  $AB$  is the chord of the arc  $AFB$ , and it is also the chord of the arc  $ACB$ .

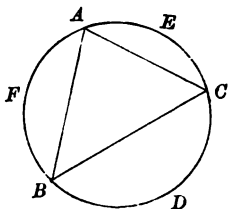
The figure formed of the arc  $BFEC$  and the chord  $BC$  is called the segment  $BFEC$  or  $BFAC$ , or more often  $BFC$  or  $BEC$  or  $BAC$ ; and the figure formed of the arc  $BDC$  and the chord  $BC$  is called the segment  $BDC$ .



**DEFINITION 3.** *The angle contained by two chords joining a point in an arc of a circle to the extremities of the arc is called an angle in the arc, and the arc is said to contain the angle.*

An angle in an arc is often spoken of as an **angle in the segment** formed by the arc and the chord of the arc, and the segment is said to **contain** the angle.

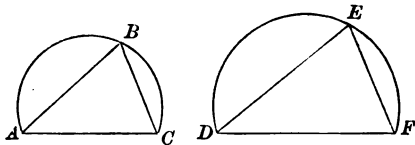
The angle  $BAC$  is said to be an angle in the arc (or in the segment)  $BFC$  and the angle  $ACB$  an angle in the arc (or in the segment)  $ADB$ ; and the arc (or the segment)  $BFC$  is said to **contain** the angle  $BAC$ , and the arc (or the segment)  $ADB$  to contain the angle  $ACB$ .



*An angle in an arc is said to stand on the arc which forms the remainder of the circle.*

The angle  $BAC$  is said to stand on the arc  $BDC$ , and the angle  $ABC$  on the arc  $AEC$ .

**DEFINITION 4.** *Arcs, which contain equal angles, are said to be similar; and likewise segments, which contain equal angles, are said to be similar.*



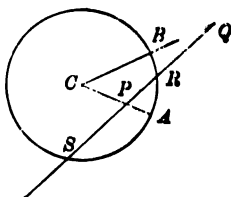
In the diagram the arcs  $ABC$ ,  $DEF$  are said to be similar, when the contained angles  $ABC$ ,  $DEF$  are equal; and also the segments  $ABC$ ,  $DEF$  are said to be similar when the contained angles  $ABC$ ,  $DEF$  are equal.

**DEFINITION 5.** *A point, whose distance from the centre of a circle is less than the radius of the circle, is said to be within the circle; and a point, whose distance from the centre of a circle is greater than the radius of the circle, is said to be without the circle.*

In the diagram the point  $P$  is within the circle, its distance  $CP$  from the centre  $C$  being less than the radius  $CA$ , and the point  $Q$  is without the circle, its distance  $CQ$  from the centre  $C$  being greater than the radius  $CB$ .

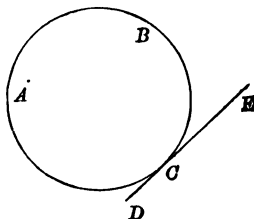
It is clear that any line drawn from a point  $P$  within a circle to a point  $Q$  without the circle must intersect the circle once at least (I. Postulate 7, page 14).

In the diagram the straight line  $PQ$  meets the circle in the points  $R$  and  $S$ , and the straight line and the circle intersect at each of those points.

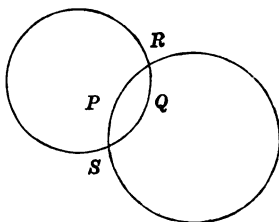


**DEFINITION 6.** *A straight line and a circle, which pass through a point but do not intersect there, are said to touch one another at the point. The straight line is called a tangent to the circle.*

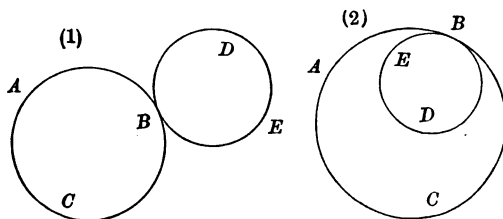
In the diagram the circle  $ABC$  and the straight line  $DCE$  pass through the point  $C$ , but do not intersect there: they touch at the point  $C$ , and  $DE$  is a tangent to the circle at the point  $C$ .



In the diagram the circles  $PRS$ ,  $QRS$  meet at the points  $R$  and  $S$ , and the circles intersect at each of those points: for instance, points on the circle  $PRS$  near  $R$  on one side of  $R$  lie *within* the circle  $QRS$  and on the other side of  $R$  *without* it.



**DEFINITION 7.** *Two circles, which pass through a point but do not intersect there, are said to touch one another at the point.*



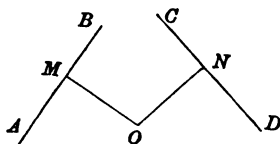
In each of the figures in the diagram the circles  $ABC$ ,  $BDE$  pass through the point  $B$ , but do not intersect there: all points on the circle  $ABC$  near  $B$  lie without the circle  $BDE$ : and all points on the circle  $BDE$  near  $B$  in figure (1) lie without the circle  $ABC$ , and in figure (2) lie within the circle  $ABC$ .

(See remarks on the contact of circles on pages 199 and 201.)

**DEFINITION 8.** *Circles which have the same point for a centre are said to be concentric.*

**DEFINITION 9.** *The perpendicular drawn to a straight line from a point is called the distance of the straight line from the point.*

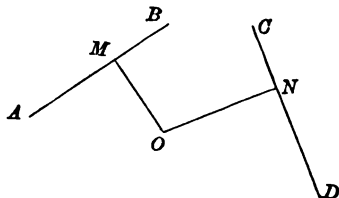
*If the distances of two straight lines from a point be equal, the straight lines are said to be equidistant from the point.*



In the diagram, if the straight lines  $OM$ ,  $ON$  be drawn from the point  $O$  perpendicular to the two straight lines  $AB$ ,  $CD$ , then  $OM$ ,  $ON$  are called the distances of the two straight lines  $AB$ ,  $CD$  from the point  $O$ .

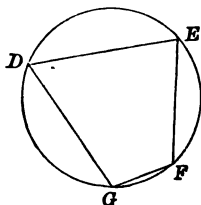
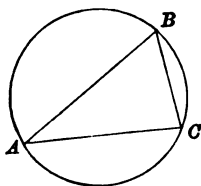
If  $OM$ ,  $ON$  be equal, the straight lines  $AB$ ,  $CD$  are said to be equidistant from the point  $O$ .

*If the distances of two straight lines from a point be unequal, the line the distance of which is the longer is said to be farther from the point, and the line the distance of which is the shorter is said to be nearer to the point.*



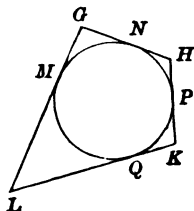
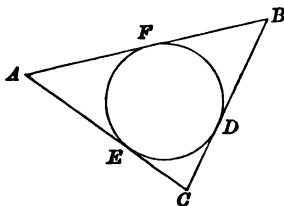
In the diagram, if the straight lines  $OM$ ,  $ON$  be drawn from the point  $O$  perpendicular to the two straight lines  $AB$ ,  $CD$ , and if  $OM$  be less than  $ON$ , then  $CD$  is said to be farther from the point  $O$  than  $AB$ , and  $AB$  is said to be nearer to the point  $O$  than  $CD$ .

**DEFINITION 10.** *If all the angular points of a rectilineal figure lie on a circle, the figure is said to be **inscribed in the circle**, and the circle is said to be **described about the figure**.*



In the diagram, if the angular points of the triangle  $ABC$  lie on the circle  $ABC$ , the triangle  $ABC$  is said to be inscribed in the circle  $ABC$ , and the circle  $ABC$  is said to be described about the triangle  $ABC$ . Similarly, if the angular points  $D, E, F, G$  of the quadrilateral  $DEFG$  lie on the circle  $DEFG$ , the quadrilateral  $DEFG$  is said to be inscribed in the circle  $DEFG$ , and the circle  $DEFG$  is said to be described about the quadrilateral  $DEFG$ .

*If all the sides of a rectilineal figure touch a circle, the figure is said to be **described about the circle**, and the circle is said to be **inscribed in the figure**.*



In the diagram, if the sides  $BC, CA, AB$  of the triangle  $ABC$  touch the circle  $DEF$  at the points  $D, E, F$  respectively, the triangle  $ABC$  is

said to be described about the circle  $DEF$ , and the circle  $DEF$  is said to be inscribed in the triangle  $ABC$ . Similarly, if the sides  $LG$ ,  $GH$ ,  $HK$ ,  $KL$  of the quadrilateral  $GHLK$  touch the circle  $MNPQ$  at the points  $M$ ,  $N$ ,  $P$ ,  $Q$  respectively, the quadrilateral  $GHLK$  is said to be described about the circle  $MNPQ$ , and the circle  $MNPQ$  is said to be inscribed in the quadrilateral  $GHLK$ .

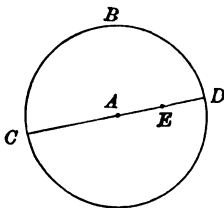
## PROPOSITION 1.

*A circle cannot have more than one centre.*

Let  $A$  be a centre of the given circle  $BCD$ :  
it is required to prove that no other point can be a centre of the circle  $BCD$ .

CONSTRUCTION. Take any point  $E$  within the circle and draw  $AE$ ,

and produce  $AE$  both ways to meet the circle, beyond  $A$  in  $C$ , and beyond  $E$  in  $D$ .



PROOF. Because  $A$  is a centre of the circle,  
 $AC$  is equal to  $AD$ . (I. Def. 22.)

Now  $EC$  is greater than  $AC$ , which is only a part of it,  
and  $ED$ , which is only a part of  $AD$ , is less than  $AD$ ;  
therefore  $EC$  is greater than  $ED$ .

But a centre of a circle is a point from which all straight  
lines drawn to the circle are equal; (I. Def. 22.)

therefore  $E$  cannot be a centre of the circle.

Similarly it can be proved that no point other than  $A$  can  
be a centre.

Wherefore, *a circle cannot have &c.*

The definition of a circle implies that the figure has a centre (I. Def. 22): it is here proved that it cannot have more than one centre: we shall therefore for the future speak of **the centre** of a circle.

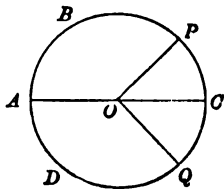


## PROPOSITION 1 A.

*A diameter bisects a circle.*

Let  $ABCD$  be a circle,  $O$  the centre and  $AOC$  a diameter: it is required to prove that the arcs  $ABC$ ,  $ADC$  are equal.

CONSTRUCTION. Draw any two radii  $OP$ ,  $OQ$  making equal angles  $POC$ ,  $QOC$  with  $OC$ . (I. Prop. 23.)



PROOF. It is possible to shift the figure  $AOCQD$  by turning it over so that  $AOC$  is not shifted, and so that the arcs  $ADC$ ,  $ABC$  lie on the same side of  $AC$ . If this be done,

because the angle  $QOC$  is equal to the angle  $POC$ ,

$OQ$  must coincide in direction with  $OP$ :

and because  $OQ$  is equal to  $OP$ ,

$Q$  must coincide with  $P$ .

Similarly it can be proved that every point on the arc  $ADC$  must coincide with some point on the arc  $ABC$ , and every point on the arc  $ABC$  with some point on the arc  $ADC$ .

Therefore the arc  $ADC$  coincides with the arc  $ABC$ , and is equal to it.

Wherefore, a diameter bisects &c.

## EXERCISES.

1. Prove by superposition that circles, which have equal radii, are equal.
2. Prove by superposition that equal circles have equal radii.
3. Two circles, which have a common centre, but whose radii are not equal, cannot meet.
4. Prove by superposition that two diameters at right angles divide a circle into four equal arcs.

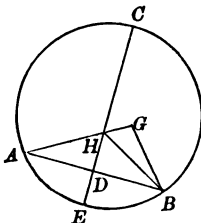
## PROPOSITION 2.

*If a straight line bisect a chord of a circle at right angles, the line passes through the centre.*

Let  $AB$  be a chord of the circle  $ABC$ , and let  $CDE$  be the straight line which bisects  $AB$  at right angles: it is required to prove that  $CDE$  passes through the centre of the circle  $ABC$ .

**CONSTRUCTION.** Take any point  $G$  not in  $CE$  and on the same side of  $CE$  as  $B$ .

Draw  $AG$  cutting  $CE$  in  $H$ , and draw  $GB$ ,  $HB$ .



**PROOF.** Because in the triangles  $ADH$ ,  $BDH$ ,

$AD$  is equal to  $BD$ ,

and  $DH$  to  $DH$ ,

and the angle  $ADH$  is equal to the angle  $BDH$ ,  
the triangles are equal in all respects; (I. Prop. 4.)

therefore  $HA$  is equal to  $HB$ .

Therefore  $GA$ , which is the sum of  $GH$ ,  $HA$ ,

is equal to the sum of  $GH$ ,  $HB$ .

And the sum of  $GH$ ,  $HB$  is greater than  $GB$ ; (I. Prop. 20.)

therefore  $GA$  is greater than  $GB$ .

But all straight lines drawn from the centre to a circle are  
equal. (I. Def. 22.)

Hence the point  $G$  cannot be the centre of the circle.

Similarly it can be proved that no point on the same side  
of  $CE$  as  $A$  can be the centre;

therefore the centre must be in  $CE$ .

Wherefore, *if a straight line &c.*

## COROLLARY 1.

*Only one chord drawn through a point within a circle which is not the centre can be bisected at the point.*

## COROLLARY 2.

*If two chords of a circle bisect each other, their point of intersection is the centre.*

## EXERCISES.

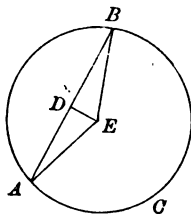
1. The straight line, which joins the middle points of two parallel chords of a circle, passes through the centre.
2. The locus of the middle points of parallel chords in a circle is a straight line.
3. Two equal parallel chords of a circle are equidistant from the centre.
4. Every parallelogram inscribed in a circle is a rectangle.
5. The diagonals of a rectangle inscribed in a circle are diameters of the circle.
6. If  $PQ, RS$  be two parallel chords of a circle and if  $PR, QS$  intersect in  $U$  and if  $PS, QR$  intersect in  $V$ , then  $UV$  passes through the centre.

## PROPOSITION 3.

*If a straight line be drawn from the centre of a circle to the middle point of a chord, which is not a diameter, it is at right angles to the chord.*

Let  $ABC$  be a circle and  $E$  its centre, and let  $D$  be the middle point of  $AB$  a chord, which is not a diameter: it is required to prove that  $ED$  is at right angles to  $AB$ .

CONSTRUCTION. Draw  $EA$ ,  $EB$ .



PROOF. Because in the triangles  $ADE$ ,  $BDE$ ,

$AD$  is equal to  $BD$ ,

$DE$  to  $DE$ ,

and  $EA$  to  $EB$ ,

the triangles are equal in all respects; (I. Prop. 8.)

therefore the angle  $ADE$  is equal to the angle  $BDE$ ,

and they are adjacent angles.

Therefore the straight lines  $ED$ ,  $AB$  are at right angles to each other. (I. Def. 11.)

Wherefore, if a straight line &c.

## EXERCISES.

1. Why are the words "which is not a diameter" inserted in enunciation of Proposition 3?

2. The straight line, which joins the middle points of two parallel chords of a circle, is at right angles to the chords.

3. Circles are described on the sides of a quadrilateral as diameters: shew that the common chord of the circles described on two adjacent sides is parallel to the common chord of the other two circles.

4. A straight line is drawn intersecting two concentric circles: prove that the portions of the straight line which are intercepted between the circles are equal.

5. A straight line cuts two concentric circles in  $P, R$ , and  $Q, S$ : prove that the rectangle  $PQ, QR$  and the rectangle  $PQ, PS$  are constant for all positions of the line.

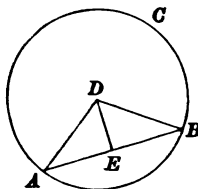
## PROPOSITION 4.

*If a straight line be drawn from the centre of a circle at right angles to a chord, it bisects the chord.*

Let  $ABC$  be a circle, and  $D$  its centre, and let the straight line  $DE$  be drawn at right angles to  $AB$  a chord, which is not a diameter\*:

it is required to prove that  $DE$  bisects  $AB$ , that is, that  $AE$  is equal to  $BE$ .

CONSTRUCTION. Draw  $DA$ ,  $DB$ .



PROOF. Because  $DB$  is equal to  $DA$ ,  
each being a radius of the circle,  
the angle  $DAB$  is equal to the angle  $DBA$ . (I. Prop. 5.)  
And the angle  $DEA$  is equal to the angle  $DEB$ ,  
each being a right angle.

Then because in the triangles  $EAD$ ,  $EBD$ ,  
the angle  $EAD$  is equal to the angle  $EBD$ ,  
and the angle  $DEA$  to the angle  $DEB$ ,  
and  $ED$ , a side opposite to a pair of equal angles, is common,  
the triangles are equal in all respects;  
(I. Prop. 26, Part 2.)

therefore  $AE$  is equal to  $BE$ .

Wherefore, *if a straight line &c.*

\* The case when the chord is a diameter requires no proof.

In Propositions 2, 3, 4 we have to deal with three properties of a straight line:

- (a) the passing through the centre of a circle,
- (b) the being at right angles to a given chord,
- (c) the bisecting the given chord.

It is proved in these propositions that, if a straight line have any two of these three properties, it necessarily has the third property.

Proposition 2 deduces (a) from (b) and (c);

Proposition 3 deduces (b) from (c) and (a);

Proposition 4 deduces (c) from (a) and (b).

#### EXERCISES.

1. Two chords are drawn through a point on a circle equally inclined to the radius drawn to the point: prove that the chords are equal.

2. If  $ABPQ$ ,  $ABRS$  be two circles and  $PR$ ,  $QS$  be any two parallel straight lines drawn through the points of section, then  $PR$ ,  $QS$  are equal.

3. If  $A$  and  $B$  be two fixed points and  $P$  move so that the perpendicular from  $A$  on  $BP$  bisects  $BP$ , the locus of  $P$  is a circle.

4. Draw through a point of intersection of two circles a straight line to make equal chords in the two circles.

5. The locus of the middle points of all chords drawn through a fixed point on a circle is a circle.

6. If two circles  $PAB$ ,  $QAB$  intersect each other at  $A$ , the locus of the middle point of a straight line  $PQ$  drawn through  $A$  is a circle.

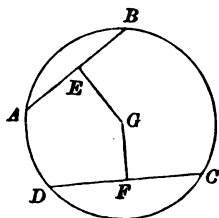
## PROPOSITION 5.

*To find the centre of a given circle.*

Let  $ABC$  be a given circle:  
it is required to find its centre.

CONSTRUCTION. Draw any two chords which are not parallel and which cut the circle in  $A, B$ , and in  $C, D$ .

Bisect  $AB$  and  $CD$  at  $E$  and  $F$ ; (I. Prop. 10.)  
and draw  $EG, FG$  at right angles to  $AB, CD$  respectively  
meeting\* at  $G$ : (I. Prop. 11.)  
then  $G$  is the centre of the circle  $ABC$ .



PROOF. Because the straight line  $EG$  bisects the chord  $AB$  at right angles,

$EG$  passes through the centre; (Prop. 2.)  
and because the straight line  $FG$  bisects the chord  $CD$  at right angles,

$FG$  passes through the centre. (Prop. 2.)  
Therefore the point  $G$ , where the two lines  $EG, FG$  meet, is the centre.

Wherefore, the centre  $G$  of the given circle  $ABC$  has been found.

\* The lines must meet. See Ex. 2, p. 51.

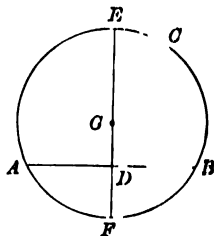


*Outline of Alternative Construction.*

Draw any chord  $AB$ , of the circle  $ABC$ .

Bisect  $AB$  in  $D$ , and draw  $EDF$  at right angles to  $AB$ , meeting the circle in  $E$  and  $F$ .

Bisect  $EF$  in  $G$ .



It may be proved that

- (1) the centre of the circle  $ABC$  is in  $EF$ ;
- (2) no other point but  $G$  can be the centre.

## EXERCISES.

1. Draw all the lines, which are wanted to find the centre of a given circle,

- (a) in the method given in the text;
- (b) when the two chords in this method meet on the circle;
- (c) in the alternative method above.

Which method requires the fewest lines?

2. Draw through a given point within a circle a chord such that it is bisected at the point.

3. Describe a circle with a given centre to cut a given circle at the extremities of a diameter.

## PROPOSITION 6.

*Every chord of a circle lies within the circle.*

Let  $AB$  be the chord joining any two points  $A, B$  on the circle  $ABC$ :

it is required to prove that any point on the chord  $AB$  between  $A$  and  $B$  is within the circle.

CONSTRUCTION. Find the centre  $D$  of the circle; (Prop. 5.) take any point  $E$  on  $AB$  between  $A$  and  $B$  and draw  $DA, DE, DB$ .

PROOF. Because in the triangle  $DAB$ ,  $DB$  is equal to  $DA$ ,

the angle  $DAB$  is equal to the angle  $DBA$ ; (I. Prop. 5.)

but the exterior angle  $DEB$  of the triangle  $DAE$  is greater than the interior opposite angle  $DAE$ ; (I. Prop. 16.) therefore the angle  $DEB$  is greater than the angle  $DBE$ .

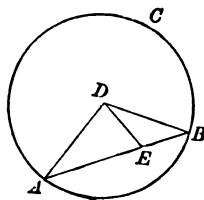
And because in the triangle  $DEB$ , the angle  $DEB$  is greater than the angle  $DBE$ ,

the side  $DB$  is greater than the side  $DE$ ; (I. Prop. 19.) that is,  $DE$  is less than  $DB$  which is a radius of the circle.

Therefore the point  $E$  is within the circle. (Def. 5.)

But  $E$  is *any* point on the chord  $AB$  between  $A$  and  $B$ , and  $AB$  is the chord joining *any* two points on the circle.

Wherefore, *every chord of a circle &c.*



## EXERCISES.

1. If a chord of a circle be produced, the produced parts lie without the circle.

2. Describe a circle which shall pass through two given points, and which shall have its radius equal to a given straight line greater in length than half the distance between the points.

How many such circles are possible?

3. Draw a straight line to cut two equal circles in  $P, Q$  and  $R, S$  so that the straight lines  $PQ, QR, RS$  may be equal.

What condition is necessary that such a straight line can be drawn?

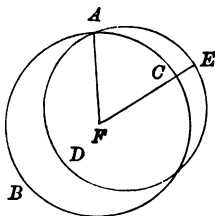
## PROPOSITION 7.

*If two circles have a common point, they cannot have the same centre.*

Let the two circles  $ABC$ ,  $ADE$  meet one another at the point  $A$ :  
it is required to prove that they cannot have the same centre.

CONSTRUCTION. Find  $F$  the centre of one of the circles  $ABC$ . (Prop. 5.)

Draw any straight line  $FCE$  meeting the circles at two distinct points  $C$  and  $E$ , and draw  $FA$ .



PROOF. Because  $F$  is the centre of the circle  $ABC$ ,  
 $FC$  is equal to  $FA$ . (I. Def. 22.)

But  $FE$  is not equal to  $FC$ ;

therefore  $FE$  is not equal to  $FA$ ;

that is, two straight lines  $FE$ ,  $FA$  drawn to the circle  $ADE$   
from the point  $F$  are not equal.

Therefore  $F$  is not the centre of the circle  $ADE$ . (I. Def. 22.)

Wherefore, if two circles &c.

## COROLLARY.

*Two concentric circles cannot have a common point.*

**DEFINITION.** *A point is said to rotate about another point, when the first point moves along a circle, of which the second point is the centre.*

*A finite straight line is said to rotate about a point, when each of its extremities moves along a circle, of which the point is the centre, while the line remains of constant length.*

*A plane figure is said to rotate about a point, when each of two points fixed in the figure moves along a circle, of which the point is the centre, while the figure remains unchanged in shape and size.*

#### ADDITIONAL PROPOSITION.

*Any finite straight line may be shifted from any one position in a plane to any other by rotation about some point in the plane.*

Let  $AB$ ,  $A'B'$  be any two positions of a finite straight line in a plane:

it is required to prove that the line can be shifted from the position

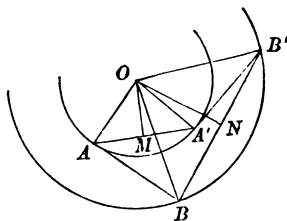
$AB$  to the position  $A'B'$  by rotation about some point in the plane.

Draw  $AA'$ ,  $BB'$  and bisect them in  $M$ ,  $N$ , and draw  $MO$ ,  $NO$  at right angles to  $AA'$ ,  $BB'$  meeting in  $O$ .

Draw  $OA$ ,  $OB$ ,  $OA'$ ,  $OB'$ .

Because in the triangles  $AOM$ ,  $A'OM$ ,  $AM$  is equal to  $A'M$ , and  $OM$  to  $OM$ , and the angle  $OMA$  to the angle  $OMA'$ ,

the triangles are equal in all respects; (I. Prop. 4.)  
therefore  $OA$  is equal to  $OA'$ .



Similarly it can be proved that  $OB$  is equal to  $OB'$ .

Again, because in the triangles  $OAB$ ,  $OA'B'$ ,  
 $OA$  is equal to  $OA'$ ,  $OB$  to  $OB'$ , and  $AB$  to  $A'B'$ ,  
 the triangles are equal in all respects; (I. Prop. 8.)  
 therefore the angle  $AOB$  is equal to the angle  $A'OB'$ ;  
 add to each the angle  $BOA'$ ;  
 then the angle  $AOA'$  is equal to the angle  $BOB'$ .

It appears therefore that the triangle  $OAB$  can be shifted into the position  $OA'B'$  by being turned in its own plane round the point  $O$  through an angle  $AOA'$  or  $BOB'$ ;  
 therefore  $AB$  can be shifted to  $A'B'$  by rotation round the point  $O$ .

# EXERCISES.

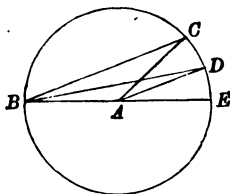
1. About what point must  $AB$  one side of a parallelogram  $ABCD$  rotate in order to take (1) the position  $CD$ , (2) the position  $DC$ ?
2. Prove that, when a straight line rotates about a point, every point in the line rotates about the point through the same angle.
3. Any triangle can be shifted from any one position to any other position, which it can occupy in the same plane without being turned over, by rotation about some point in the plane.
4. Prove that, when a plane figure rotates about a point, every point in the figure rotates about the point through the same angle.
5. Describe an equilateral triangle of which one angular point is given and the others lie on two given straight lines.  
 How many solutions are there?
6. Construct an equilateral triangle, one of whose angular points is given and the other two lie one on each of two given circles.  
 Find the limits of the position of the given point which admit of a possible solution.
7. Construct a square to have one vertex at a fixed point and two opposite vertices on two given straight lines.

## PROPOSITION 8. PART 1.

*Of all straight lines drawn to a circle from a point on the circle, the line which is a diameter is the greatest; and of any two others, the one which subtends the greater angle at the centre is the greater.*

Let  $CDE$  be a given circle,  $A$  its centre, and  $B$  any point on the circle; let  $BAE$  be a diameter, and let  $BC$ ,  $BD$  be any other two straight lines drawn from  $B$  to the circle, and of the angles  $BAC$ ,  $BAD$  subtended by  $BC$ ,  $BD$  at  $A$  let the angle  $BAD$  be the greater:

it is required to prove that  $BE$  is greater than  $BD$ , and  $BD$  greater than  $BC$ .



PROOF. Because  $AE$  is equal to  $AD$ ,  
 therefore  $BE$ , which is the sum of  $BA$ ,  $AE$ ,  
 is equal to the sum of  $BA$ ,  $AD$ :  
 but the sum of  $BA$ ,  $AD$  is greater than  $BD$ ; (I. Prop. 20.)  
 therefore  $BE$  is greater than  $BD$ .

Next, because in the triangles  $BAD$ ,  $BAC$ ,  
 $AD$  is equal to  $AC$ ,  
 and  $BA$  to  $BA$ ,

and the angle  $BAD$  is greater than the angle  $BAC$ ,  
 therefore  $BD$  is greater than  $BC$ . (I. Prop. 24.)

Wherefore, of all straight lines &c.

## COROLLARY.

*A diameter is the greatest chord of a circle.*

**EXERCISES.**

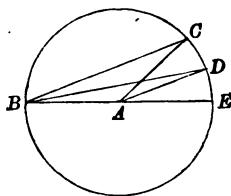
1. If two chords of a circle subtend equal angles at the centre, they are equal.
2. If two chords of a circle be equal, they subtend equal angles at the centre.
3. Of any two chords in a circle the one which subtends the greater angle at the centre is the greater.

## PROPOSITION 8. PART 1.

*Of all straight lines drawn to a circle from a point on the circle, the line which is a diameter is the greatest; and of any two others, the one which subtends the greater angle at the centre is the greater.*

Let  $CDE$  be a given circle,  $A$  its centre, and  $B$  any point on the circle; let  $BAE$  be a diameter, and let  $BC$ ,  $BD$  be any other two straight lines drawn from  $B$  to the circle, and of the angles  $BAC$ ,  $BAD$  subtended by  $BC$ ,  $BD$  at  $A$  let the angle  $BAD$  be the greater:

it is required to prove that  $BE$  is greater than  $BD$ , and  $BD$  greater than  $BC$ .



PROOF. Because  $AE$  is equal to  $AD$ ,  
 therefore  $BE$ , which is the sum of  $BA$ ,  $AE$ ,  
 is equal to the sum of  $BA$ ,  $AD$ :  
 but the sum of  $BA$ ,  $AD$  is greater than  $BD$ ; (I. Prop. 20.)  
 therefore  $BE$  is greater than  $BD$ .  
 Next, because in the triangles  $BAD$ ,  $BAC$ ,  
 $AD$  is equal to  $AC$ ,  
 and  $BA$  to  $BA$ ,  
 and the angle  $BAD$  is greater than the angle  $BAC$ ,  
 therefore  $BD$  is greater than  $BC$ . (I. Prop. 24.)  
 Wherefore, of all straight lines &c.

## COROLLARY.

*A diameter is the greatest chord of a circle.*



normal points  
= centre.

normal points  
are equal.

normality of one  
the other side

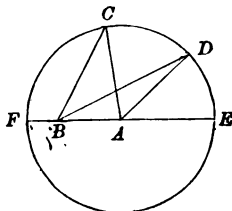
## PROPOSITION 8. PART 2.

*Of all straight lines drawn to a circle from an internal point not the centre, the one which passes through the centre is the greatest; and the one which when produced passes through the centre is the least; and of any two others, the one which subtends the greater angle at the centre is the greater.*

Let  $CDE$  be a given circle,  $A$  its centre and  $B$  any other internal point; let  $BA$  produced beyond  $A$  cut the circle in  $E$ , and produced beyond  $B$  in  $F$ , and let  $BC$ ,  $BD$  be any other two straight lines drawn from  $B$  to the circle, and of the angles  $BAC$ ,  $BAD$  subtended by  $BC$ ,  $BD$  at  $A$  let the angle  $BAD$  be the greater:

it is required to prove that  $BE$  is greater than  $BD$ ,

$BD$  greater than  $BC$ , and  $BC$  greater than  $BF$ .



PROOF. Because  $AE$  is equal to  $AD$ ,  
therefore  $BE$ , which is the sum of  $BA$ ,  $AE$ , is equal to  
the sum of  $BA$ ,  $AD$ ;

but the sum of  $BA$ ,  $AD$  is greater than  $BD$ ; (I. Prop. 20.)  
therefore  $BE$  is greater than  $BD$ .

Next, because in the triangles  $BAD$ ,  $BAC$ ,  
 $AD$  is equal to  $AC$ ,

$BA$  to  $BA$ ,

and the angle  $BAD$  is greater than the angle  $BAC$ ,  
therefore  $BD$  is greater than  $BC$ . (I. Prop. 24.)

Again, because the sum of  $BC$ ,  $BA$  is greater than  $AC$ ,  
(I. Prop. 20.)

and  $AC$  is equal to  $AF$ , which is the sum of  $BF$ ,  $BA$ ,  
therefore the sum of  $BC$ ,  $BA$  is greater than the sum of  
 $BF$ ,  $BA$ .

Therefore  $BC$  is greater than  $BF$ .

Wherefore, of all straight lines &c.

REPORT

Whereas the Commission has been authorized by the  
the following: That the Commission has been authorized  
to conduct a study of the various factors which  
influence the economic development of the various  
countries of the world, and to report thereon to the  
Commission, and to the Secretary of the Commission,  
and to the President of the United States, and to the  
Congress of the United States, and to the people of the  
United States, and to the world.

Whereas the Commission has been authorized by the  
the following: That the Commission has been authorized

to conduct a study of the various factors which  
influence the economic development of the various  
countries of the world, and to report thereon to the  
Commission, and to the Secretary of the Commission,  
and to the President of the United States, and to the  
Congress of the United States, and to the people of the  
United States, and to the world.

REPORT

I find the various factors which influence the economic  
development of the various countries of the world to be

the following: That the Commission has been authorized  
to conduct a study of the various factors which  
influence the economic development of the various  
countries of the world, and to report thereon to the  
Commission, and to the Secretary of the Commission,  
and to the President of the United States, and to the  
Congress of the United States, and to the people of the  
United States, and to the world.

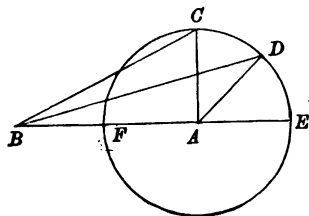
Under these conditions the Commission has been authorized

## PROPOSITION 8. PART 3.

*Of all straight lines drawn to a circle from an external point, the one which passes through the centre is the greatest, and the one which when produced passes through the centre is the least ; and of any two others, the one which subtends the greater angle at the centre is the greater.*

Let  $CDE$  be a given circle,  $A$  its centre and  $B$  a given external point ; let  $BA$  cut the circle in  $F$  and let  $BA$  produced cut the circle in  $E$ , and let  $BD, BC$  be any other two straight lines drawn from  $B$  to the circle, and of the angles  $BAC, BAD$  subtended by  $BC, BD$  at  $A$  let the angle  $BAD$  be the greater :

it is required to prove that  $BE$  is greater than  $BD$ ,  
 $BD$  greater than  $BC$ , and  $BC$  greater than  $BF$ .



**PROOF.** Because  $AE$  is equal to  $AD$ ,  
 therefore  $BE$ , which is the sum of  $BA, AE$ ,  
 is equal to the sum of  $BA, AD$  :  
 but the sum of  $BA, AD$  is greater than  $BD$  ; (I. Prop. 20.)  
 therefore  $BE$  is greater than  $BD$ .

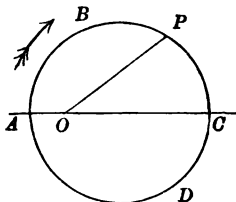
Next, because in the triangles  $BAD, BAC$ ,  
 $AD$  is equal to  $AC$ ,  
 and  $BA$  to  $BA$ ,

and the angle  $BAD$  is greater than the angle  $BAC$ ,  
 therefore  $BD$  is greater than  $BC$ . (I. Prop. 24.)  
 Again, because the sum of  $BC, CA$  is greater than  $BA$ ,  
 (I. Prop. 20.)

which is the sum of  $BF, FA$ ,  
 and because  $CA$  is equal to  $FA$ ,  
 therefore  $BC$  is greater than  $BF$ .

Wherefore, of all straight lines &c.

We conclude from the results of the several Parts of Proposition 8 that, if  $O$  be a fixed point on the diameter  $AC$  of a circle  $ABCD$  nearer to  $A$  than to  $C$  and  $P$  be a point which is capable of motion along the circumference in the direction represented by the arrow, while  $P$  is moving along the arc  $ABC$  from  $A$  to  $C$  the distance  $OP$  increases continuously from  $OA$  to  $OC$ , and while  $P$  is moving along the arc  $CDA$  from  $C$  to  $A$ , the distance  $OP$  decreases continuously from  $OC$  to  $OA$ .



We say therefore that  $OC$  is a *maximum* value of  $OP$ , and  $OA$  is a *minimum* value. (See remarks on page 55.)

It may be observed here that, if  $P$  travel round the circle any number of times, it passes  $C$  and  $A$  alternately. It appears therefore that here *maximum* and *minimum* values occur alternately.

The occurrence of *maximum* and *minimum* values alternately is true generally in the case of quantities which vary continuously, i.e. quantities whose magnitude changes without suffering any abrupt changes.

### EXERCISES.

1. Find the shortest distance between two points one on each of two circles which do not meet.
2.  $A$  and  $B$  are two fixed points; it is required to find a point  $P$  on a given circle, so that the sum of the squares on  $AP$  and  $BP$  may be the least possible.

Under what conditions is the solution indeterminate?

## PROPOSITION 9.

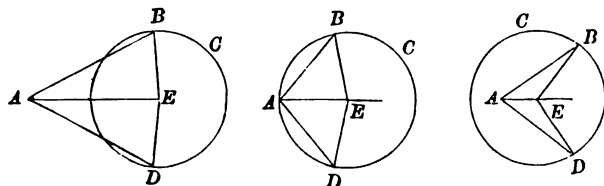
*From a point not the centre not more than two equal straight lines can be drawn to a circle, one on each side of the straight line drawn from that point to the centre.*

Let  $A$  be a given point, and  $BCD$  a given circle, and let  $AB$ ,  $AD$  be two equal straight lines drawn from  $A$  to the circle:

it is required to prove that no other straight line equal to  $AB$  or  $AD$  can be drawn from  $A$  to the circle.

CONSTRUCTION. Find  $E$  the centre of the circle;  
(Prop. 5.)

draw  $EA$ ,  $EB$ ,  $ED$ .



PROOF. Take  $C$  any point of the circle on the same side of  $AE$  as  $AB$ .

Because  $B$  and  $C$  are equidistant from  $E$ , they cannot be equidistant from  $A$ . (I. Prop. 7.) Therefore there cannot be another straight line equal to  $AB$  drawn from  $A$  to the circle on the same side of  $AE$  as  $AB$ .

Similarly it can be proved that there cannot be another straight line equal to  $AD$  drawn from  $A$  to the circle on the same side of  $AE$  as  $AD$ .

Wherefore, from a point &c.

## COROLLARY 1.

*If from a point three equal straight lines can be drawn to a circle, that point is the centre.*

## COROLLARY 2.

*Two circles cannot meet in more than two points.*

If the straight line  $AB$  drawn to the circle from a point  $A$  not the centre be in the same straight line as the centre of the circle, no other straight line can be drawn to the circle from the point  $A$  equal to  $AB$ . The line  $AB$  is in this case either a *maximum* or a *minimum* among the straight lines drawn from the point  $A$  to the circle; it is a *maximum*, if  $B$  be at the further extremity of the diameter through  $A$ , and a *minimum*, if  $B$  be at the nearer extremity.

## EXERCISES.

1. If from any point within a circle two straight lines be drawn to the circumference making equal angles with the straight line joining the point and the centre, the lines are equal in length.
2. Construct an equilateral triangle, having two of its vertices on a given circle and the third at a given point within the circle.
3. Construct a square having one vertex at a given point and two opposite vertices on a given circle.

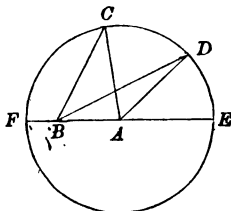
## PROPOSITION 8. PART 2.

*Of all straight lines drawn to a circle from an internal point not the centre, the one which passes through the centre is the greatest; and the one which when produced passes through the centre is the least; and of any two others, the one which subtends the greater angle at the centre is the greater.*

Let  $CDE$  be a given circle,  $A$  its centre and  $B$  any other internal point; let  $BA$  produced beyond  $A$  cut the circle in  $E$ , and produced beyond  $B$  in  $F$ , and let  $BC$ ,  $BD$  be any other two straight lines drawn from  $B$  to the circle, and of the angles  $BAC$ ,  $BAD$  subtended by  $BC$ ,  $BD$  at  $A$  let the angle  $BAD$  be the greater:

it is required to prove that  $BE$  is greater than  $BD$ ,

$BD$  greater than  $BC$ , and  $BC$  greater than  $BF$ .



PROOF. Because  $AE$  is equal to  $AD$ ,  
therefore  $BE$ , which is the sum of  $BA$ ,  $AE$ , is equal to  
the sum of  $BA$ ,  $AD$ ;

but the sum of  $BA$ ,  $AD$  is greater than  $BD$ ; (I. Prop. 20.)  
therefore  $BE$  is greater than  $BD$ .

Next, because in the triangles  $BAD$ ,  $BAC$ ,

$AD$  is equal to  $AC$ ,

$BA$  to  $BA$ ,

and the angle  $BAD$  is greater than the angle  $BAC$ ,

therefore  $BD$  is greater than  $BC$ . (I. Prop. 24.)

Again, because the sum of  $BC$ ,  $BA$  is greater than  $AC$ ,

(I. Prop. 20.)

and  $AC$  is equal to  $AF$ , which is the sum of  $BF$ ,  $BA$ ,  
therefore the sum of  $BC$ ,  $BA$  is greater than the sum of  
 $BF$ ,  $BA$ .

Therefore  $BC$  is greater than  $BF$ .

Wherefore, of all straight lines &c.



**EXERCISES.**

1. If two straight lines drawn to a circle from an internal point not the centre be equal, they subtend equal angles at the centre.
2. If two straight lines drawn to a circle from an internal point not the centre subtend equal angles at the centre, they are equal.
3. If each of two equal straight lines have one extremity on one of two concentric circles and the other extremity on the other, the lines subtend equal angles at the common centre.

## PROPOSITION 11.

*If two circles meet at a point, which lies in the same straight line as their centres and is between the centres, the circles touch at that point, and each circle lies without the other.*

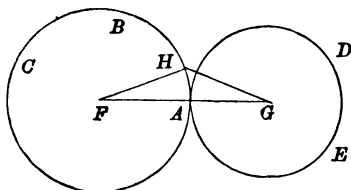
Let  $ABC$ ,  $ADE$  be two circles meeting at a point  $A$ , which is in the same straight line as their centres, and is between the centres :

it is required to prove that the circles touch at  $A$ , and that each circle lies without the other.

CONSTRUCTION. Find the centres  $F$ ,  $G$  of the circles  $ABC$ ,  $ADE$ ; (Prop. 5.)

draw  $FG$ , which by the hypothesis passes through  $A$ .

Take any point  $H$  on the circle  $ABC$ ,  
and draw  $FH$ ,  $HG$ .



PROOF. Because the sum of  $FH$ ,  $HG$  is greater than  $FG$ , (I. Prop. 20.)

that is, greater than the sum of  $FA$ ,  $AG$ ,

and  $FH$  is equal to  $FA$ ,

therefore  $HG$  is greater than  $AG$ .

But  $AG$  is a radius of the circle  $ADE$ ;

therefore the distance of the point  $H$  from the centre of the circle  $ADE$  is greater than the radius, and  $H$  therefore is without the circle  $ADE$ . (Def. 5.)

Therefore every point on the circle  $ABC$  except  $A$  lies without the circle  $ADE$ .

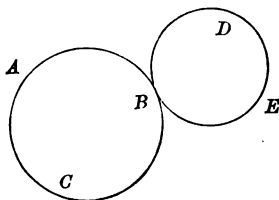
Therefore the circles touch at  $A$ . (Def. 7.)

Similarly it can be proved that every point on the circle  $ADE$  except  $A$  lies without the circle  $ABC$ .

Wherefore, if two circles &c.

When one circle touches another circle which lies without it, the first circle is said to have **external contact** with the second circle.

In the diagram each of the circles  $ABC$ ,  $BDE$  has external contact with the other at the point  $B$ .



## EXERCISES.

1. If the distance between the centres of two circles be greater than the sum of their radii, each circle lies without the other.

2. Prove that in all cases the greatest distance between two points one on each of two given circles is greater than the distance between the centres by the sum of the radii.

3. Of two equal circles of given radius, which touch externally at  $P$ , one touches  $OX$  and the other touches  $OY$ , where  $OX$ ,  $OY$  are two given straight lines at right angles to each other: prove that the locus of  $P$  is an equal circle.

Shew that there are four such loci.

4. If two equal circles touch, every straight line drawn through the point of contact will make equal chords in the two circles.

5. Given two concentric circles, draw a chord of the outer which shall be trisected by the inner circle.

6. Three circles touch one another externally at the points  $A$ ,  $B$ ,  $C$ ; the straight lines  $AB$ ,  $AC$  are produced to cut the circle  $BC$  at  $D$  and  $E$ : shew that  $DE$  is a diameter of  $BC$ , and is parallel to the straight line joining the centres of the other circles.

## PROPOSITION 12.

*If two circles meet at a point, which lies in the same straight line as their centres and is not between the centres, the circles touch at that point, and one of the circles lies within the other.*

Let  $ABC$ ,  $ADE$  be two circles meeting at a point  $A$ , which is in the same straight line as their centres and is not between the centres :

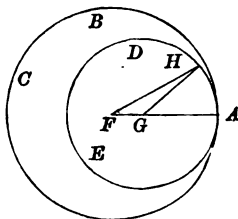
it is required to prove that the circles touch at  $A$ , and that one of the circles lies within the other.

CONSTRUCTION. Find the centres  $F$ ,  $G$  of the circles  $ABC$ ,  $ADE$  ;

(Prop. 5.)

draw  $FG$ , and produce  $FG$  which by the hypothesis passes through  $A$ . Let  $ADE$  be the circle whose centre  $G$  is the nearer to  $A$ .

Take  $H$  any point on the circle  $ADE$ , and draw  $FH$ ,  $HG$ .



PROOF. Because  $GA$  is equal to  $GH$ ,  $FA$ , which is the sum of  $FG$ ,  $GA$ , is equal to the sum of  $FG$ ,  $GH$  ;

but the sum of  $FG$ ,  $GH$  is greater than  $FH$  ; (I. Prop. 20.)  
therefore  $FA$  is greater than  $FH$ .

But  $FA$  is a radius of the circle  $ABC$  ;  
therefore the distance of the point  $H$  from the centre of the circle  $ABC$  is less than the radius, and  $H$  therefore is within the circle  $ABC$  ;  
therefore every point on the circle  $ADE$  except  $A$  lies within the circle  $ABC$ .

Therefore the circles touch at  $A$ .

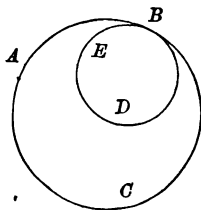
(Def. 7.)

Wherefore, if two circles &c.

When one circle touches another circle, which lies within it, the first circle is said to have **internal contact** with the second circle.

Two circles can have external contact with each other, but two circles cannot have internal contact with each other. If one circle have *internal* contact with another circle, the second circle has *external* contact with the first circle.

In the diagram the circle  $ABC$  has **internal contact** with the circle  $BDE$ , but the circle  $BDE$  has **external contact** with the circle  $ABC$  at the point  $B$ .



## EXERCISES.

1. Describe a circle passing through a given point and touching a given circle at a given point.
2. If in any two given circles which touch one another, there be drawn two parallel diameters, the point of contact and an extremity of each diameter, lie in the same straight line.
3. Describe a circle which shall touch a given circle, have its centre in a given straight line, and pass through a given point in the given straight line.
4. Describe a circle of given radius to pass through a given point and to touch a given circle.

What conditions are necessary that a solution may be possible?

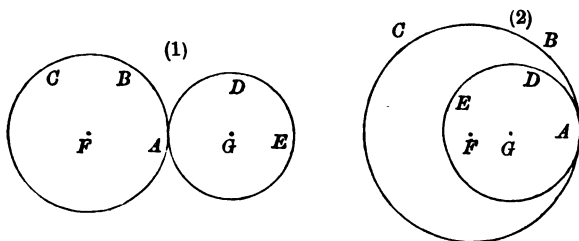
## PROPOSITION 13.

*If two circles have a point of contact, they do not meet at any other point.*

Let  $ABC$ ,  $ADE$  be two circles which touch at the point  $A$  :

it is required to prove that the circles do not meet at any other point.

CONSTRUCTION. Find  $F$ ,  $G$  the centres of the circles  $ABC$ ,  $ADE$ . (Prop. 5.)



PROOF. Because the circles  $ABC$ ,  $ADE$  touch, the point of contact  $A$  must lie in the straight line  $FG$  or in  $FG$  produced. (Prop. 10, Coroll.)

First (fig. 1) let the point  $A$  lie in  $FG$  :  
then because the circles  $ABC$ ,  $ADE$  meet at a point  $A$  in the same straight line  $FG$  as their centres and between the centres,  
each circle lies without the other. (Prop. 11.)

Secondly (fig. 2) let the point  $A$  lie in  $FG$  produced :  
then because the circles  $ABC$ ,  $ADE$  meet at a point  $A$  in the same straight line  $FG$  as their centres and not between the centres,  
one circle lies within the other. (Prop. 12.)  
Therefore in neither case can the circles meet at any point other than  $A$ .

Wherefore, if two circles &c.

We infer as a result of Propositions 9—13 that two circles must be such that they either

- (a) intersect in two distinct points,
- or (b) touch at one point, which is in the straight line joining the centres,
- or (c) do not meet.

#### EXERCISES.

1. What is the greatest number of contacts which may exist among (1) three, (2) four circles?
2. Describe three circles to have their centres at three given points, and to touch each other in pairs.
3. Into how many parts will three circles divide a plane? Distinguish between the different cases which may occur, when the circles intersect or touch.

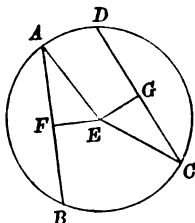
## PROPOSITION 14. PART 1.

*Chords of a circle, which are equal, are equidistant from the centre.*

Let  $AB, CD$  be two equal chords of the circle  $ABCD$ : it is required to prove that  $AB, CD$  are equidistant from the centre.

CONSTRUCTION. Find  $E$  the centre of the circle  $ABCD$ ; (Prop. 5.)  
and from  $E$  draw  $EF, EG$  at right angles to  $AB, CD$ . (I. Prop. 12.)

Draw  $EA, EC$ .



PROOF. Because the straight line  $EF$  is drawn through the centre of the circle at right angles to the chord  $AB$ , it bisects it; (Prop. 4.)

that is,  $AB$  is double of  $AF$ .

Similarly it can be proved that  $CD$  is double of  $CG$ .

But  $AB$  is equal to  $CD$ ;  
therefore  $AF$  is equal to  $CG$ .

Next, because the angles  $AFE, CGE$  are right angles, the square on  $AE$  is equal to the sum of the squares on  $AF, FE$ ,  
and the square on  $CE$  is equal to the sum of the squares on  $CG, GE$ . (I. Prop. 47.)

And because  $AE$  is equal to  $CE$ ,  
the square on  $AE$  is equal to the square on  $CE$ .  
Therefore the sum of the squares on  $AF, FE$  is equal to the sum of the squares on  $CG, GE$ .

Because  $AF$  is equal to  $CG$ ,  
the square on  $AF$  is equal to the square on  $CG$ ;  
therefore the square on  $FE$  is equal to the square on  $GE$ .



Therefore  $FE$  is equal to  $GE$ ,  
that is,  $AB, CD$  are distant from the centre.

(Def. 9.)

Wherefore, *chords of a circle &c.*

#### EXERCISES.

1. Chords of a circle, which are equal, subtend equal angles at the centre.
2. Chords of a circle, which subtend equal angles at the centre, are equidistant from the centre.

## PROPOSITION 14. PART 2.

*Chords of a circle, which are equidistant from the centre, are equal.*

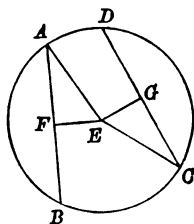
Let  $AB$ ,  $CD$  be two chords of the circle  $ABCD$  equidistant from the centre :

it is required to prove that  $AB$  is equal to  $CD$ .

CONSTRUCTION. Find  $E$  the centre of the circle  $ABCD$  ;  
(Prop. 5.)

and from  $E$  draw  $EF$ ,  $EG$  at right angles to  $AB$ ,  $CD$ .  
(I. Prop. 12.)

Draw  $EA$ ,  $EC$ .



PROOF. Because the straight line  $EF$  is drawn through the centre of the circle at right angles to the chord  $AB$ , it bisects it ;  
(Prop. 4.)

that is,  $AB$  is double of  $AF$ .

Similarly it may be proved that  $CD$  is double of  $CG$ .

Next, because the angles  $AFE$ ,  $CGE$  are right angles, the square on  $AE$  is equal to the sum of the squares on  $AF$ ,  $FE$ ,  
and the square on  $CE$  is equal to the sum of the squares on  $CG$ ,  $GE$ .  
(I. Prop. 47.)

And because  $AE$  is equal to  $CE$ ,

the square on  $AE$  is equal to the square on  $CE$ .

Therefore the sum of the squares on  $AF$ ,  $FE$  is equal to the sum of the squares on  $CG$ ,  $GE$ .

Because  $EF$  is equal to  $EG$ ,

the square on  $EF$  is equal to the square on  $EG$  ;

therefore the square on  $AF$  is equal to the square on  $CG$  ;  
therefore  $AF$  is equal to  $CG$ .

And it has been proved that  $AB$  is double of  $AF$ , and  $CD$  of  $CG$ .

Therefore  $AB$  is equal to  $CD$ .

Wherefore, *chords of a circle &c.*

Parts 1 and 2 of Proposition 14 are the converses of each other.

#### EXERCISES.

1. In a circle chords, which are equidistant from the centre, subtend equal angles at the centre.
2. In a circle chords, which subtend equal angles at the centre, are equal.

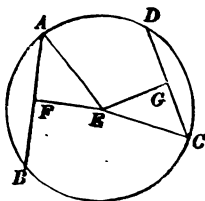
## PROPOSITION 15. PART 1.

*Of any two chords of a circle the one which is the greater is the nearer to the centre.*

Let  $AB$ ,  $CD$  be two chords of the circle  $ABCD$ , of which  $AB$  is greater than  $CD$ :  
it is required to prove that  $AB$  is nearer to the centre than  $CD$ .

CONSTRUCTION. Find  $E$  the centre of the circle  $ABCD$ ;  
(Prop. 5.)  
and from  $E$  draw  $EF$ ,  $EG$  at right angles to  $AB$ ,  $CD$ .  
(I. Prop. 12.)

Draw  $EA$ ,  $EC$ .



PROOF. Because the straight line  $EF$  is drawn through the centre at right angles to the chord  $AB$ ,

$AF$  is equal to  $FB$ , (Prop. 4.)  
and  $AB$  is double of  $AF$ .

Similarly it can be proved that  $CD$  is double of  $CG$ .

But  $AB$  is greater than  $CD$ ;  
therefore  $AF$  is greater than  $CG$ .

Next, because the angles  $AFE$ ,  $CGE$  are right angles,  
the square on  $AE$  is equal to the sum of the squares on  
 $AF$ ,  $FE$ ,  
and the square on  $CE$  is equal to the sum of the squares on  
 $CG$ ,  $GE$ . (I. Prop. 47.)

And because  $AE$  is equal to  $CE$ ,  
the square on  $AE$  is equal to the square on  $CE$ .  
Therefore the sum of the squares on  $AF$ ,  $FE$  is equal to  
the sum of the squares on  $CG$ ,  $GE$ ;

Because  $AF$  is greater than  $CG$ ,  
the square on  $AF$  is greater than the square on  $CG$ ;  
therefore the square on  $FE$  is less than the square on  $GE$ .

Therefore  $FE$  is less than  $GE$ ,  
that is,  $AB$  is nearer to the centre than  $CD$ .

Wherefore, of *any two chords &c.*

**EXERCISES.**

1. Prove that every straight line, which makes equal chords in two equal circles, is parallel to the straight line joining the centres or passes through the middle point of that line.
2. Find the shortest chord which can be drawn through a given point within a circle.

## PROPOSITION 15. PART 2.

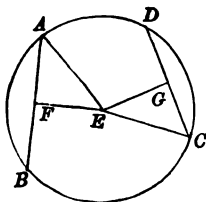
*Of any two chords of a circle the one which is the nearer to the centre is the greater.*

Let  $AB, CD$  be two chords of the circle  $ABCD$ , of which  $AB$  is nearer to the centre than  $CD$ :

it is required to prove that  $AB$  is greater than  $CD$ .

CONSTRUCTION. Find  $E$  the centre of the circle  $ABCD$ ;  
(Prop. 5.)  
and from  $E$  draw  $EF, EG$  at right angles to  $AB, CD$ .  
(I. Prop. 12.)

Draw  $EA, EC$ .



PROOF. Because the straight line  $EF$  is drawn through the centre at right angles to the chord  $AB$ ,

$AF$  is equal to  $FB$ , (Prop. 4.)  
and  $AB$  is double of  $AF$ .

Similarly it can be proved that  $CD$  is double of  $CG$ .

Next, because the angles  $AFE, CGE$  are right angles, the square on  $AE$  is equal to the sum of the squares on  $AF, FE$ ,

and the square on  $CE$  is equal to the sum of the squares on  $CG, GE$ . (I. Prop. 47.)

And because  $AE$  is equal to  $CE$ ,  
the square on  $AE$  is equal to the square on  $CE$ .

Therefore the sum of the squares on  $AF, FE$  is equal to the sum of the squares on  $CG, GE$ .

Because  $EF$  is less than  $EG$ ,  
the square on  $EF$  is less than the square on  $EG$ ;

therefore the square on  $AF$  is greater than the square on  $CG$ ;

therefore  $AF$  is greater than  $CG$ .

Therefore  $AB$  is greater than  $CD$ .

Wherefore, of any two chords &c.

Parts 1 and 2 of Proposition 15 are the converses of each other.

#### EXERCISES.

1. Of any two chords of a circle the nearer to the centre subtends the greater angle at the centre.

2. Draw through a given point a straight line to make equal chords in two given equal circles.

Discuss the number of possible solutions in the different cases which may occur.

## PROPOSITION 16.

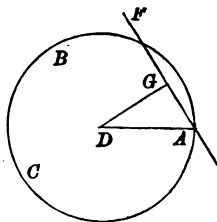
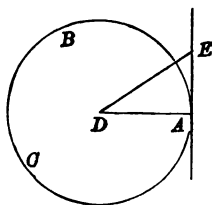
*The straight line drawn through a point on a circle at right angles to the radius touches the circle, and every other straight line drawn through the point cuts the circle.*

Let  $ABC$  be a circle, of which  $D$  is the centre, and let  $AE$  be a straight line drawn through  $A$  at right angles to the radius  $AD$ ; and let  $AF$  be any other straight line drawn through  $A$ :

it is required to prove that  $AE$  touches the circle, and that  $AF$  cuts the circle.

CONSTRUCTION. Take any point  $E$  on  $AE$ , and draw  $DE$ , and from  $D$  draw  $DG$  at right angles to  $AF$ .

(I. Prop. 12.)



PROOF. Because in the triangle  $DAE$ ,

the angle  $DAE$  is a right angle,

the angle  $DEA$  is less than a right angle; (I. Prop. 17.)

therefore the angle  $DAE$  is greater than the angle  $DEA$ ;

therefore  $DE$  is greater than  $DA$ . (I. Prop. 19.)

Therefore the distance of the point  $E$  from the centre is greater than the radius, and  $E$  therefore is without the circle. (Def. 5.)

Similarly it can be proved that every point on  $AE$  except  $A$  is without the circle.

Therefore  $AE$  touches the circle  $ABC$  at  $A$ . (Def. 6.)

Next because in the triangle  $DGA$ ,

the angle  $DGA$  is a right angle,

the angle  $DAG$  is less than a right angle; (I. Prop. 17.)

therefore the angle  $DAG$  is less than the angle  $DGA$ ;

therefore  $DG$  is less than  $DA$ . (I. Prop. 19.)



Therefore the distance of the point  $G$  from the centre is less than the radius, and therefore  $G$  is within the circle. (Def. 5.)

Therefore the straight line  $AF$  cuts the circle  $ABC$  \*.

Wherefore, the straight line &c.

We infer as a result of Proposition 16 that a straight line and a circle must be such that they either

- (a) intersect in two distinct points,
- or (b) touch at one point,
- or (c) do not meet.

### EXERCISES.

1. A point  $B$  is taken on a circle whose centre is  $C$ ;  $PA$  a tangent at any point  $P$  meets  $CB$  produced at  $A$ , and  $PD$  is drawn perpendicular to  $CB$ : prove that  $PB$  bisects the angle  $APD$ .

2. Describe a circle to have its centre on a given straight line, to pass through a given point on that line and to touch another given straight line.

3. Describe a circle to pass through a given point and to touch a given straight line at a given point.

4. If  $AC$  be a diameter of a circle  $ABC$ , and  $AP$  be drawn perpendicular to the tangent at  $B$ ,  $AB$  bisects the angle  $CAP$ .

5. Prove that although no straight line can be drawn to pass between a circle and its tangent, yet any number of circles can be described to do so.

6. Circles, which have a common tangent at a point, touch each other.

7. Prove that the angle between a tangent to a circle and a chord drawn from the point of contact is half of the angle subtended at the centre by the chord.

\* A straight line which cuts a circle is often called a secant.

## PROPOSITION 17.

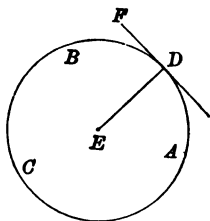
*Through a given point to draw a tangent to a given circle.*

Let  $ABC$  be a given circle, and  $D$  a given point :  
it is required through  $D$  to draw a tangent to the circle  $ABC$ .

First, let the point  $D$  be on the circle.

CONSTRUCTION. Find the centre  $E$ ;  
(Prop. 5.)

draw  $ED$ , and draw  $DF$  at right angles to  $DE$ ;  
(I. Prop. 11.)  
then  $DF$  is a tangent drawn as required.

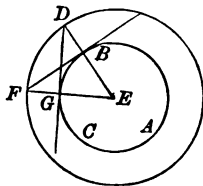


(Prop. 16.)

Secondly, let the point  $D$  be outside the circle.

CONSTRUCTION. Find the centre  $E$ ;  
(Prop. 5.)

draw  $ED$ , cutting the circle  $ABC$  between  $E$  and  $D$  in  $B$ , and draw  $BF$  at right angles to  $EB$ , (I. Prop. 11.) and with  $E$  as centre and  $ED$  as radius describe a circle cutting  $BF$  in  $F$ . Draw  $EF$ , cutting the circle  $ABC$  between  $E$  and  $F$  in  $G$ , and draw  $DG$  :  
then  $DG$  is a tangent drawn as required.



PROOF. Because in the triangles  $DEG$ ,  $FEB$ ,  
 $DE$  is equal to  $FE$ ,  $EG$  to  $EB$ , and the angle  $DEG$  equal to the angle  $FEB$ ,

the triangles are equal in all respects ; (I. Prop. 4.)

therefore the angle  $DGE$  is equal to the angle  $FBE$ .

But the angle  $FBE$  is a right angle ;

therefore the angle  $DGE$  is a right angle ;

and because the straight line  $GD$  is drawn through the point  $G$  on the circle  $ABC$  at right angles to  $GE$  the radius,  $GD$  touches the circle. (Prop. 16.)

Wherefore, *through a given point  $D$  a tangent has been drawn to the circle  $ABC$ .*

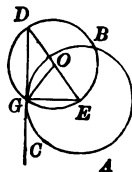
*Outline of Alternative Construction.*

Find  $E$  the centre. (Prop. 5.)

Draw  $ED$ , and bisect it in  $O$ . (I. Prop. 10.)

With  $O$  as centre and  $OD$  as radius describe a circle, cutting the circle  $ABC$  in  $G$ .

Draw  $DG$ ,  $OG$ ,  $EG$ .



It may be proved that

- (1) the angle  $OGD$  is equal to the angle  $GDO$ ,
- (2) the angle  $OGE$  is equal to the angle  $GEO$ ,
- and (3) the angle  $EGD$  is a right angle,

and hence that  $GD$  is a tangent to the circle  $ABC$  at  $G$ .

Both the construction in the Proposition and the alternative construction point out that two and only two tangents can be drawn to a circle through an external point, one through a point on the circle, and none through an internal point.

When there is no danger of ambiguity the length of the straight line drawn from an external point to touch a circle which is intercepted between that point and the point of contact is often spoken of as the *tangent* from the point to the circle.

EXERCISES.

1. The two tangents drawn to a circle from an external point are equal.
2. Draw a tangent to a given circle to be parallel to a given straight line.
3. Find in a given straight line a point such that the tangent drawn from it to a given circle may be equal to a given straight line.
4. The greater the distance of an external point is from the centre of a circle, the smaller is the inclination of the two tangents which can be drawn from it.

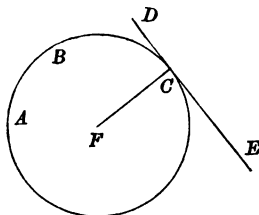
## PROPOSITION 18.

*If a straight line touch a circle, the radius drawn to the point of contact is at right angles to the line.*

Let the straight line  $DE$  touch the circle  $ABC$  at the point  $C$ :

it is required to prove that the radius drawn to the point  $C$  is at right angles to  $DCE$ .

CONSTRUCTION. Find  $F$  the centre of the circle; (Prop. 5.) and draw  $FC$ .



PROOF. If  $DCE$  were not at right angles to  $CF$ ,  
 $DCE$  would cut the circle (Prop. 16);  
but it does not:  
therefore  $DCE$  is at right angles to  $CF$ .  
Wherefore, *if a straight line &c.*

## THE TANGENT AS THE LIMIT OF THE SECANT.

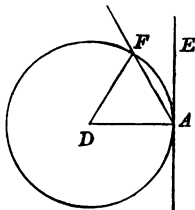
Let  $AF$  be a straight line cutting a given circle at a given point  $A$ , and again at a second point  $F$ .

Let  $D$  be the centre of the circle, and  $AE$  the tangent at  $A$ . Draw  $DA$ ,  $DF$ .

The angle  $FAE$  is equal to half of the angle  $ADF$ . (See Ex. 7, p. 213.)

Hence the smaller the angle  $ADF$  is, or the smaller the chord  $AF$ , (Prop. 8, Part 1.) the smaller is the angle  $FAE$ .

Now because we can take the point  $F$  as close to  $A$  as we like, we can make the angle  $ADF$ , and therefore also the angle  $FAE$ , as small as we like. Hence we can make the straight line  $AF$  deviate as little as we please from coincidence with  $AE$ .



We express this fact by saying that, the tangent  $AE$  is the limit of the secant  $AF$ , when  $F$  moves up close to  $A$ .

*This definition of a tangent to a curve as the limit of the secant through the point is one which admits of application to curves of all kinds.*

## EXERCISES.

1. Through a given point draw a straight line so that the chord which is intercepted on it by a given circle is equal to a given straight line.
2. Two circles are concentric: prove that all chords of the outer circle which touch the inner are equal.
3. If two tangents be drawn to a circle from an external point, the chord joining the points of contact is bisected at right angles by the straight line joining the centre and the external point.



## ADDITIONAL PROPOSITION.

To draw a common tangent to two given circles.

Let  $A, B$  be the centres of two given circles, which we will call for shortness the (A) circle and the (B) circle :

Let the radius of the (A) circle be greater than the radius of the (B) circle.

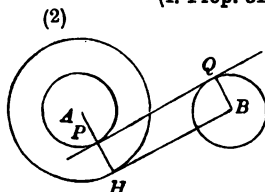
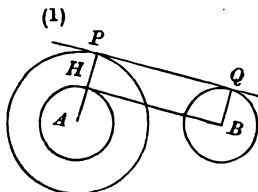
With  $A$  for centre and the difference (fig. 1) or the sum (fig. 2) of the radii for radius describe a circle,

and from  $B$  draw  $BH$  a tangent to it. (Prop. 17.)

Draw  $AH$  and let  $AH$  produced (fig. 1) or  $AH$  (fig. 2) cut the (A) circle in  $P$ .

Through  $P, B$  draw  $PQ, BQ$  parallel to  $HB, AH$  respectively.

(I. Prop. 31.)



Because  $HPQB$  is a parallelogram, (Constr.)

$BQ$  is equal to  $HP$ , (I. Prop. 34.)

which is equal to the radius of the (B) circle. (Constr.)

Therefore the point  $Q$  is on the (B) circle.

Again, because  $BH$  is a tangent at  $H$ ,

the angle  $PHB$  is a right angle ; (Prop. 18.)

therefore the parallelogram  $HPQB$  is a rectangle ; (I. Def. 19.)

therefore the angles at  $P$  and  $Q$  are right angles, (I. Prop. 29, Coroll.)

and  $PQ$  is a tangent to the (A) and (B) circles at  $P$  and  $Q$

respectively. (Prop. 16.)

## EXERCISES.

1. Prove that four common tangents can be drawn to two circles which are external to each other.

2. How many common tangents can be drawn to two intersecting circles?

3. Is it possible that two circles can have one and only one common tangent?

4. Draw a straight line so that the chords which are intercepted on it by two given circles are equal to two given straight lines.

## PROPOSITION 20.

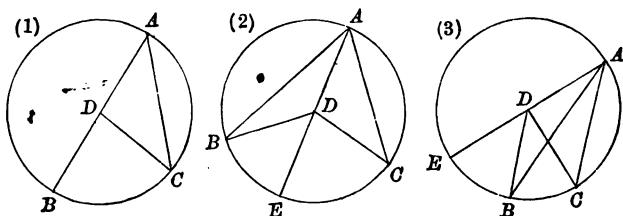
*The angle which an arc of a circle subtends at the centre is double of the angle which the arc subtends at the circumference.*

Let  $ABC$  be a circle, of which  $BC$  is an arc, and let  $BDC$ ,  $BAC$  be angles subtended by the arc  $BC$  at the centre  $D$ , and at the circumference :

it is required to prove that the angle  $BDC$  is double of the angle  $BAC$ .

First, (fig. 1) let the centre  $D$  lie on  $AB$ , one of the lines which contain the angle  $BAC$ .

CONSTRUCTION. Draw  $DC$ .



PROOF. Because  $DA$  is equal to  $DC$ ,  
the angle  $DCA$  is equal to the angle  $DAC$ ; (I. Prop. 5.)  
therefore the sum of the angles  $DAC$ ,  $DCA$  is double of the angle  $DAC$ .

But the angle  $BDC$  is equal to the sum of the angles  $DAC$ ,  $DCA$ ; (I. Prop. 32.)  
therefore the angle  $BDC$  is double of the angle  $DAC$ .

Next, let the centre  $D$  lie within (fig. 2) or without (fig. 3) the angle  $BAC$ .

CONSTRUCTION. Draw  $AD$  and produce it to meet the circle in  $E$ .

PROOF. It follows from the first case, that the angle  $EDC$  is double of the angle  $EAC$ , and that the angle  $EDB$  is double of the angle  $EAB$ ;



therefore in (fig. 2) the sum of the angles  $EDC$ ,  $EDB$  is double of the sum of the angles  $EAC$ ,  $EAB$ , and in (fig. 3) the difference of the angles  $EDC$ ,  $EDB$  is double of the difference of the angles  $EAC$ ,  $EAB$ ; therefore in all cases the angle  $BDC$  is double of the angle  $BAC$ .

Wherefore, *the angle which an arc &c.*

In the diagram of Proposition 20 in each of the figures the angle  $BDC$  is double of the angle  $BAC$ . Now it is easily seen that although in figures (1) and (3) the angle  $BAC$  is restricted to values less than a right angle, and the angle  $BDC$  in consequence to values less than two right angles, in figure (2) the angle  $BAC$  is restricted only to values less than two right angles and the angle  $BDC$  in consequence only to values less than four right angles. It appears therefore that, if we wish not to destroy the generality of the theorem of Proposition 20, we must allow our definition of an angle to include angles which are equal to two or greater than two right angles; there is nothing inconsistent with a strict adherence to Euclid's methods in doing so.

#### EXERCISES.

1. Two circles, whose centres are  $A$  and  $D$ , touch externally at  $E$ : a third circle, whose centre is  $B$ , touches them internally at  $C$  and  $F$ : prove that the angle  $ADB$  is double of the angle  $ECF$ .

2. If  $AB$  be a fixed diameter and  $DE$  an arc of constant length in a fixed circle, and the straight lines  $AE$ ,  $BD$  intersect at  $P$ , the angle  $APB$  is constant.

3. If  $ABC$  be a triangle inscribed in a circle and the angle  $BAC$  be bisected by  $AD$ , which meets the circle in  $D$ , then the diameter through  $D$  will bisect  $BC$  at right angles.

4.  $AB$  is a diameter and  $PQ$  any chord of a circle cutting  $AB$  within the circle, and  $AL$  is drawn perpendicular to  $PQ$ . Prove that the angle  $LAB$  is equal to the sum of the angles  $PAB$ ,  $QAB$ .

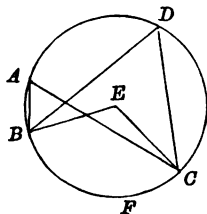
## PROPOSITION 21:

*Angles in the same arc of a circle are equal.*

Let  $ABCD$  be a circle, and  $BAC$ ,  $BDC$  be two angles in the same arc  $BADC$ :

it is required to prove that the angles  $BAC$ ,  $BDC$  are equal.

CONSTRUCTION. Find the centre  $E$ ; (Prop. 5.)  
and draw  $EB$ ,  $EC$ .



PROOF. Because the angle which the arc  $BFC$  of the circle subtends at the centre is double of the angle which it subtends at the circumference,

the angle  $BEC$  is double of the angle  $BAC$ ,  
and also the angle  $BEC$  is double of the angle  $BDC$ ;  
(Prop. 20.)

therefore the angle  $BAC$  is equal to the angle  $BDC$ .

Wherefore, *angles in the same arc &c.*

COROLLARY. *If a straight line joining two points subtend equal angles at two other points on the same side of the line, the four points lie on a circle.*

Let the straight line  $BC$  subtend equal angles at the two points  $A$ ,  $D$  on the same side of  $BC$ .

If a circle be described about the triangle  $BAC^*$ , the

\* That it is possible to describe a circle through the three vertices of a triangle appears in the Additional Proposition on page 58.

circle must cut  $BD$  again at some point not on the same side of  $AC$  as  $B$ . (Prop. 6.)

Now take  $H$  any point but  $D$  in  $BD$  or  $BD$  produced and draw  $HD$ .

Then the angle  $BHC$  cannot be equal to the angle  $BDC$ ,  
(I. Prop. 16.)

and therefore cannot be equal to the angle  $BAC$ .

But angles in the same arc of a circle are equal. (Prop. 21.)

Therefore the circle  $BAC$  cannot meet  $BD$  in  $H$ ;  
that is, it must meet it in  $D$ .

In some books in the proof of Proposition 21, the result of Proposition 20 is quoted as if it were true only in the case of arcs greater than a semicircle: that is, as if the angle  $BEC$ , which the arc  $BFC$  subtends at the centre, were restricted to magnitudes less than two right angles. The general truth of the theorem is then deduced as a consequence.

We leave this deduction to the student as an exercise.

### EXERCISES.

1. The locus of a point at which a given straight line subtends a constant angle is an arc of a circle.
2. If of three concurrent straight lines inclined at given angles to one another two pass through two fixed points, the third also passes through a third fixed point.
3. If two sides of a triangle of constant shape and size pass through two fixed points, the third always touches a fixed circle.
4. If two sides of a triangle of constant shape and size always touch two fixed circles, the third side always touches a fixed circle.
5. If  $ABC$  be an equilateral triangle described in a circle whose centre is  $O$ , and if  $AO$  produced meet the circle in  $D$ , then  $OD$ ,  $BC$  bisect each other.
6. Two circles  $ADB$ ,  $ACB$  intersect in points  $A$  and  $B$ . Through  $A$  any chord  $DAC$  is drawn, and  $BC$ ,  $BD$  are joined, and the angle  $DBC$  is internally bisected by a line  $BE$  which meets  $DC$  in  $E$ . Shew that  $E$  lies on a fixed circle.
7. If  $ABC$  be an isosceles triangle on the base  $BC$ , inscribed in a circle, and  $P$ ,  $Q$  be points on the arcs  $AC$ ,  $AB$  respectively of the circle such that  $AQ$  is parallel to  $BP$ , then  $CQ$  is parallel to  $AP$ .
8. If the diagonals of a quadrilateral inscribed in a circle be at right angles, the perpendicular from their intersection on any side bisects the opposite side.

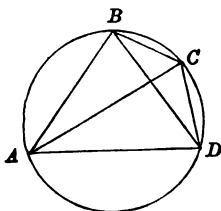
## PROPOSITION 22.

*The sum of two opposite angles of a convex quadrilateral inscribed in a circle is equal to two right angles.*

Let  $ABCD$  be a quadrilateral inscribed in the circle  $ABCD$ :

it is required to prove that the sum of the angles  $ABC$ ,  $ADC$  is equal to two right angles, and that the sum of the angles  $BAD$ ,  $BCD$  is equal to two right angles.

CONSTRUCTION. Draw  $AC$ ,  $BD$ .



PROOF. Because the angles  $BCA$ ,  $BDA$  are in the same arc  $BDA$ ,

the angle  $BCA$  is equal to the angle  $BDA$ ; (Prop. 21.) and because the angles  $CAB$ ,  $CDB$  are in the same arc  $CDA$ ,

the angle  $CAB$  is equal to the angle  $CDB$ . (Prop. 21.) Therefore the sum of the angles  $BCA$ ,  $CAB$  is equal to the sum of the angles  $BDA$ ,  $CDB$ , that is, to the angle  $ADC$ .

To each of these equals add the angle  $ABC$ : then the sum of the angles  $ABC$ ,  $BCA$ ,  $CAB$  is equal to the sum of the angles  $ABC$ ,  $ADC$ .

But because the angles  $ABC$ ,  $BCA$ ,  $CAB$  are the angles of a triangle, their sum is equal to two right angles.

(I. Prop. 32.)

Therefore the sum of the angles  $ABC$ ,  $ADC$  is equal to two right angles.

Similarly it can be proved that the sum of the angles  $BAD$ ,  $BCD$  is equal to two right angles.

Wherefore, the sum of two opposite angles &c.

**COROLLARY.** *If the sum of two opposite angles of a convex quadrilateral be equal to two right angles, the vertices of the quadrilateral lie on a circle.*

Let  $ABCD$  be a convex quadrilateral in which the sum of the angles  $BAD$ ,  $BCD$  is equal to two right angles.

If a circle be described about the triangle  $BAD$ , the circle must cut  $AC$  again in some point not on the same side of  $BD$  as  $A$ . (Prop. 6.)

Now take  $H$  any point but  $C$  in  $AC$  or  $AC$  produced and draw  $HB$ ,  $HD$ .

Then the angle  $BHD$  cannot be equal to the angle  $BCD$ . (I. Prop. 21.)

Therefore the sum of the angles  $BAD$ ,  $BHD$  cannot be equal to two right angles.

But the sum of two opposite angles of a convex quadrilateral inscribed in a circle is equal to two right angles. (Prop. 22.)

Therefore the circle  $BAD$  cannot meet  $AC$  in  $H$ ;  
that is, it must meet it in  $C$ .

### EXERCISES.

1. If the sides  $AB$ ,  $DC$  of a quadrilateral  $ABCD$  inscribed in a circle be produced to meet at  $E$ , the triangles  $AEC$ ,  $BED$  are equiangular to one another.

2. A triangle is inscribed in a circle: shew that the sum of the angles in the three segments exterior to the triangle is equal to four right angles.

3. If  $PQRS$ ,  $pqr$  be two circles, and  $PprR$ ,  $QqsS$  be chords such that  $P$ ,  $p$ ,  $q$ ,  $Q$  lie on a circle, then  $R$ ,  $r$ ,  $s$ ,  $S$  lie on a circle.

4. If any two consecutive sides of a convex hexagon inscribed in a circle be respectively parallel to their opposite sides, the remaining sides are parallel to each other.

5. If any arc of a circle described on the side  $BC$  of a triangle  $ABC$  cut  $BA$ ,  $CA$  produced if necessary in  $P$  and  $Q$ ,  $PQ$  is always parallel to a fixed straight line.

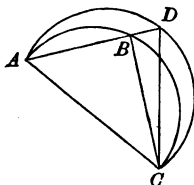
6.  $E$  is a point on one of the diagonals  $AC$  of a parallelogram  $ABCD$ . Circles are described about  $DEA$  and  $BEC$ . Shew that  $BD$  passes through the other point of intersection of the circles.

## PROPOSITION 23.

*Two arcs of circles, which have a common chord and are on the same side of it, cannot be similar unless they are coincident.*

Let  $ABC$ ,  $ADC$  be two arcs of circles, which have a common chord  $AC$ , and are on the same side of it: it is required to prove that  $ABC$ ,  $ADC$  cannot be similar arcs, unless they are coincident.

CONSTRUCTION. Draw through  $A$ , one of the extremities of the chord  $AB$ , any straight line  $ABD$  to meet the arcs in  $B$ ,  $D$ ; and draw  $CB$ ,  $CD$ .



PROOF. If the points  $B$ ,  $D$  do not coincide, one of the angles  $ABC$ ,  $ADC$  is an exterior angle and the other an interior angle of the triangle  $BCD$ ; therefore the angle  $ABC$  is not equal to the angle  $ADC$ ,  
(I. Prop. 16.)  
and therefore the arc  $ABC$  is not similar to the arc  $ADC$ .  
(Def. 4.)

It has now been proved that, if any straight line  $ABD$  meet the arcs in two points  $B$  and  $D$  which are not coincident,

the arcs cannot be similar.

Therefore, if the arcs be similar, every straight line drawn through  $A$  must meet the arcs in two coincident points, that is, the arcs  $ABC$ ,  $ADC$  must coincide.

Wherefore, *two arcs of circles &c.*

**EXERCISES.**

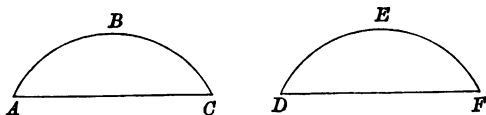
1. If on opposite sides of the same straight line there be two arcs of circles, which contain supplementary angles, the arcs are parts of the same circle.
2. Prove that, if two circles have three points in common, the circles are coincident.

## PROPOSITION 24.

*Similar arcs of circles, which have equal chords, are equal.*

Let  $ABC$ ,  $DEF$  be two similar arcs of circles, which have equal chords  $AC$ ,  $DF$ :

it is required to prove that the arcs  $ABC$ ,  $DEF$  are equal.



PROOF. Because the chords  $AC$ ,  $DF$  are equal, it is possible to shift the figure  $ABC$ , so that  $AC$  coincides with  $DF$ ,  $A$  with  $D$ , and  $C$  with  $F$ , (1. Test of Equality, page 5)

and so that the arcs  $ABC$ ,  $DEF$  are on the same side of  $DF$ .

If this be done,

the arc  $ABC$  must coincide with the arc  $DEF$ ,

for the arcs  $ABC$ ,  $DEF$  are similar,

and two arcs of circles, which have a common chord, and are on the same side of it, cannot be similar unless they coincide. (Prop. 23.)

Therefore the arcs  $ABC$ ,  $DEF$  are equal.

Wherefore, *similar arcs of circles &c.*



## EXERCISES.

1. If  $D$  be a point in the side  $BC$  of a triangle  $ABC$  whose sides  $AB, AC$  are equal, the circles described about the triangles  $ABD, ACD$  are equal.
2. Find a point  $P$  within an equilateral triangle  $ABC$ , such that the circles described about the triangles  $PBC, PCA, PAB$  may be all equal.
3. Find a point  $P$  in the plane of a triangle  $ABC$  such that the circles described about the triangles  $PBC, PCA, PAB$  may be equal.

## PROPOSITION 25.

*To find the centre of the circle, of which a given arc is a part.*

Let  $ABC$  be a given arc:  
it is required to find the centre of the circle, of which the arc  $ABC$  is a part.

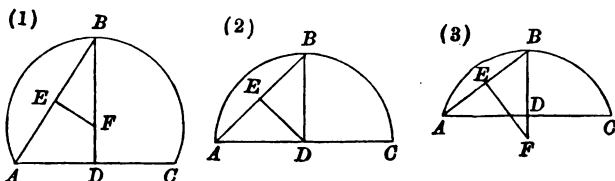
CONSTRUCTION. Draw  $AC$ , and bisect it at  $D$ .

(I. Prop. 10.)

At  $D$  draw  $DB$  at right angles to  $AC$  cutting the arc at  $B$ .

(I. Prop. 11.)

Draw  $AB$ , bisect it at  $E$ , and at  $E$  draw  $EF$  at right angles to  $AB$  meeting  $BD$  or  $BD$  produced at  $F^*$ ;  
then  $F$  is the centre required.



PROOF. Because  $DB$  bisects the chord  $AC$  at right angles,

$DB$  passes through the centre; (Prop. 2.)

and because  $EF$  bisects the chord  $AB$  at right angles,

$EF$  passes through the centre. (Prop. 2.)

Now two straight lines cannot intersect in more than one point. (I. Post. 1.)

Therefore  $F$ , the point of intersection of  $BD$  and  $EF$ ,  
is the centre.

Wherefore, the centre of the circle, of which the given arc  $ABC$  is a part, has been found.

\* The lines must meet, see Ex. 2, p. 51. In figure (2)  $F$  coincides with  $D$ .

## PROPOSITION 25 A.

*Equal circles have equal radii.*

If two circles be equal, it is possible to shift one of them so as to coincide with the other. (I. Def. 21, page 13.)

Let this be done.

Then, because a circle cannot have more than one centre, (Prop. 1.)  
the centres of the two coincident circles must be coincident:  
and therefore all the radii of both circles are equal.

Wherefore, *equal circles &c.*

## EXERCISES.

1. Having given two arcs of circles, shew how to find whether they are parts of the same circle.
2. Having given two arcs of circles, find whether they are parts of concentric circles.
3. Having given two arcs of circles, find whether one circle lies wholly within the other.

## PROPOSITION 26.

*In equal circles the arcs, on which equal angles at the centres stand, are equal; and the arcs, on which equal angles at the circumferences stand, are equal.*

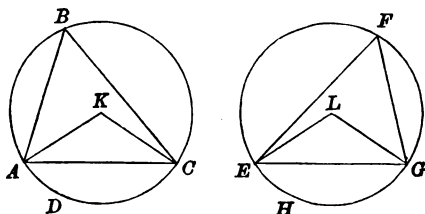
Let  $ABCD$ ,  $EFGH$  be two equal circles, and let  $AKC$ ,  $ELG$  be two angles at the centres standing on the arcs  $ADC$ ,  $EHG$ , and let  $ABC$ ,  $EFG$  be two angles at the circumferences standing on the same arcs: and let

(1) the angles  $AKC$ ,  $ELG$  be equal,

(2) the angles  $ABC$ ,  $EFG$  be equal:

it is required in either case to prove that the arcs  $ADC$ ,  $EHG$  are equal.

CONSTRUCTION. Draw  $AC$ ,  $EG$ .



PROOF. Because the angle  $AKC$  is double of the angle  $ABC$ , (Prop. 20.)

and the angle  $ELG$  is double of the angle  $EFG$ , (Prop. 20.)

in case (1) because the angles  $AKC$ ,  $ELG$  are equal,

the angles  $ABC$ ,  $EFG$  are equal,

and in case (2) because the angles  $ABC$ ,  $EFG$  are equal,

the angles  $AKC$ ,  $ELG$  are equal.

Now because the circles are equal,

their radii  $AK$ ,  $KC$ ,  $EL$ ,  $LG$  are equal. (Prop. 25 A.)

Therefore in both cases (1) and (2),

because in the triangles  $AKC$ ,  $ELG$ ,

$AK$  is equal to  $EL$ ,

$KC$  to  $LG$ ,

and the angle  $AKC$  to the angle  $ELG$ ,

the triangles are equal in all respects; (I. Prop. 4.)

therefore  $AC$  is equal to  $EG$ .

And because the arcs  $ABC$ ,  $EFG$ , which contain equal angles, have equal chords  $AC$ ,  $EG$ ,  
                   the arcs  $ABC$ ,  $EFG$  are equal: (Prop. 24.)  
                   but the circles  $ABCD$ ,  $EFGH$  are equal;  
                   therefore the remaining arcs  $ADC$ ,  $EHG$  are equal.  
 Wherefore, in equal circles &c.

**COROLLARY.** *In the same circle the arcs, on which equal angles at the centres stand, are equal; and the arcs, on which equal angles at the circumferences stand, are equal.*

## EXERCISES.

1. If  $PQ$ ,  $RS$  be a pair of parallel chords in a circle, then the arcs  $PS$ ,  $QR$  are equal, and the arcs  $PR$ ,  $QS$  are equal.
2. A quadrilateral is inscribed in a circle, and two opposite angles are bisected by straight lines meeting the circumference in  $P$  and  $Q$ ; prove that  $PQ$  is a diameter.
3. If through  $P$  any point on one of two circles, which intersect in  $A$  and  $B$ , the straight lines  $PA$ ,  $PB$  be drawn and produced if necessary to cut the other circle in  $Q$  and  $R$ , the arc  $QR$  is of constant length.
4. The internal bisectors of the vertical angles of all triangles, on the same base and on the same side of it, which have equal vertical angles, pass through one fixed point and the external bisectors through another fixed point.
5. If through one of the points of intersection of two equal circles a straight line be drawn terminated by the circles, the straight lines joining its extremities with the other point of intersection are equal.

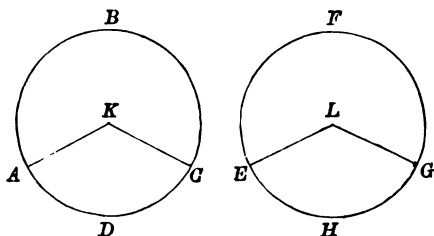
## PROPOSITION 27.

*In equal circles, angles contained by arcs, which are of equal length, are equal.*

Let  $ABCD$ ,  $EFGH$  be equal circles, and let  $ABC$ ,  $EFG$  be arcs of equal length:

it is required to prove that the arcs  $ABC$ ,  $EFG$  contain equal angles.

CONSTRUCTION. Find the centres  $K$ ,  $L$  of the circles  $ABCD$ ,  $EFGH$ ,  
(Prop. 5.)  
and draw  $AK$ ,  $KC$ ,  $EL$ ,  $LG$ .



PROOF. Because the circles  $ABCD$ ,  $EFGH$  are equal, their radii  $AK$ ,  $KC$ ,  $EL$ ,  $LG$  are equal. (Prop. 25 A.) And because  $AK$  is equal to  $EL$ , it is possible to shift the figure  $ABCDK$  so that  $A$  coincides with  $E$ , and  $K$  with  $L$ , and so that the parts of the arcs  $ABC$ ,  $EFG$  near  $E$  are on the same side of  $EL$ .

If this be done, because the radii of the circles are equal and their centres coincide,

the circles must coincide;

and because the circles coincide, and the parts of the arcs  $ABC$ ,  $EFG$  near  $E$  are on the same side of  $EL$ , those parts of the arcs coincide;

and because the arcs are of equal length and have one extremity common,

therefore the other extremity must be common,  
that is, the point  $C$  must coincide with the point  $G$ .

Therefore the arc  $ABC$  coincides with the arc  $EFG$  ;  
and angles in the two arcs are then angles in the same arc  
and therefore equal. (Prop. 21.)

Wherefore, *in equal circles &c.*

COROLLARY. *In the same circle, angles contained by arcs, which are of equal length, are equal.*

#### EXERCISES.

1. The straight lines joining the extremities of two equal arcs of a circle are parallel or are equal.

Can they be both parallel and equal?

2. The straight lines bisecting any angle of a quadrilateral inscribed in a circle and the opposite exterior angle, meet on the circle.

3. If from any point on a circle a chord and a tangent be drawn, the perpendiculars on them from the middle point of either of the arcs subtended by the chord are equal to one another.

## PROPOSITION 28.

*In equal circles, arcs cut off by chords, which are equal to one another, are of equal length, the greater equal to the greater and the less equal to the less.*

Let  $ABCD$ ,  $EFGH$  be equal circles, and let  $AC$ ,  $EG$  be equal chords which cut off the two greater arcs  $ABC$ ,  $EFG$ , and the two less arcs  $ADC$ ,  $EHG$  :

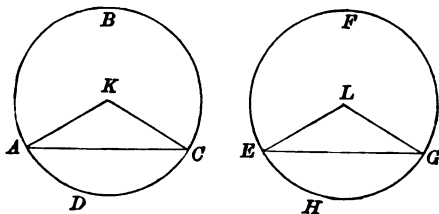
it is required to prove that the arcs  $ABC$ ,  $EFG$  are of equal length,

and that the arcs  $ADC$ ,  $EHG$  are of equal length.

CONSTRUCTION. Find  $K$ ,  $L$ , the centres of the circles  $ABCD$ ,  $EFGH$ ,

(Prop. 5.)

and draw  $AK$ ,  $KC$ ,  $EL$ ,  $LG$ .



PROOF. Because the circles are equal,  
their radii  $AK$ ,  $KC$ ,  $EL$ ,  $LG$  are equal ; (Prop. 25 A.)  
and because in the triangles  $AKC$ ,  $ELG$ ,

$KA$  is equal to  $LE$ ,

$KC$  to  $LG$ ,

and  $AC$  to  $EG$ ,

the triangles are equal in all respects ; (I. Prop. 8.)

therefore the angle  $AKC$  is equal to the angle  $ELG$ .

But in equal circles the arcs, on which equal angles at the centres stand, are equal ; (Prop. 26.)

therefore the arc  $ADC$  is equal to the arc  $EHG$ .

But the circle  $ABCD$  is equal to the circle  $EFGH$ ;

therefore the arc  $ABC$  is equal to the arc  $EFG$ .

Wherefore, in equal circles &c.



**COROLLARY.** *In the same circle, arcs cut off by chords, which are equal to one another, are of equal length, the greater equal to the greater and the less equal to the less.*

### EXERCISES.

1. A triangle is turned about its vertex till one of the sides passing through the vertex is in the same straight line as the other previously was. Prove that the line joining the vertex with the intersection of the two positions of the base, produced if necessary, bisects the angle between these two positions.

2. Find a point on one of two given equal circles, such that, if from it two tangents be drawn to the other circle, the chord joining the points of contact is equal to the chord of the first circle formed by joining its points of intersection with the two tangents produced.

Determine the conditions of the possibility of a solution of the problem.

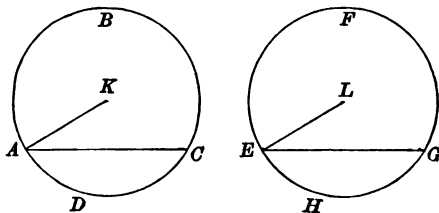
## PROPOSITION 29.

*In equal circles, chords, by which arcs of equal length are subtended, are equal.*

Let  $ABCD$ ,  $EFGH$  be equal circles, and let  $AC$ ,  $EG$  be chords by which  $ADC$ ,  $EHG$ , arcs of equal length are subtended :

it is required to prove that the chords  $AC$ ,  $EG$  are equal.

CONSTRUCTION. Find  $K$ ,  $L$ , the centres of the circles  $ABCD$ ,  $EFGH$ , (Prop. 5.)  
and draw  $AK$ ,  $EL$ .



PROOF. Because the circles  $ABCD$ ,  $EFGH$  are equal, their radii  $AK$ ,  $EL$  are equal. (Prop. 25 A.)  
And because  $AK$  is equal to  $EL$ , it is possible to shift the figure  $ABCDK$  so that  $A$  coincides with  $E$ , and  $K$  with  $L$ , and so that the parts of the arcs  $ABC$ ,  $EFG$  near  $E$  are on the same side of  $EL$ .

If this be done, because the radii of the circles are equal and their centres coincide,

the circles must coincide;

and because the circles coincide, and the parts of the arcs  $ABC$ ,  $EFG$  near  $E$  are on the same side of  $EL$ , those parts of the arcs coincide;

and because the arcs are of equal length and have one extremity common,

therefore the other extremity must be common,  
that is, the point  $C$  must coincide with the point  $G$ .

Therefore the chord  $AC$  coincides with the chord  $EG$   
and is equal to it.

Wherefore, *in equal circles &c.*

**COROLLARY.** *In the same circle, chords, by which arcs of equal length are subtended, are equal.*

#### EXERCISES.

1. If the diagonals of a quadrilateral inscribed in a circle bisect one another, the diagonals are diameters.

2. If two chords  $AP$ ,  $AQ$  of a circle intersect at a constant angle at a fixed point  $A$  on the circle, the chord  $PQ$  always touches a concentric circle.

3. Two triangles are inscribed in a circle: if two sides of the one be parallel to two sides of the other, the third sides are equal.

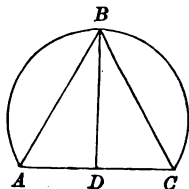
Is it necessary that they are parallel?

## PROPOSITION 30.

*To bisect a given arc of a circle.*

Let  $ABC$  be the given arc:  
it is required to bisect it.

CONSTRUCTION. Draw  $AC$ ;  
bisect it at  $D$ ; (I. Prop. 10.)  
at  $D$  draw  $DB$  at right angles to  $AC$  meeting the arc at  $B$ :  
(I. Prop. 11.)  
the arc  $ABC$  is bisected as required at the point  $B$ .  
Draw  $AB$ ,  $BC$ .



PROOF. Because in the triangles  $ADB$ ,  $CDB$ ,  
 $AD$  is equal to  $CD$ ,  
and  $DB$  to  $DB$ ,

and the angle  $ADB$  to the angle  $CDB$ ,  
the triangles are equal in all respects; (I. Prop. 4.)  
therefore  $AB$  is equal to  $CB$ .

But arcs cut off by equal chords are equal, the greater equal  
to the greater, and the less equal to the less;  
(Prop. 28, Coroll.)

and because  $BD$ , if produced, is a diameter, (Prop. 2.)  
each of the arcs  $AB$ ,  $CB$  is less than a semicircle,  
and therefore the arc  $AB$  is the smaller of the two arcs cut  
off by the chord  $AB$ , and the arc  $CB$  the smaller of those  
cut off by the chord  $CB$ ;

therefore the arc  $AB$  is equal to the arc  $CB$ .

Wherefore, *the given arc  $ABC$  is bisected at  $B$ .*

## EXERCISES.

1. Find the triangle of maximum area which can be inscribed in a given circle having a given chord for one side.
2. Prove that the triangle of maximum area inscribed in a circle is equilateral.
3. The greatest quadrilateral which can be inscribed in a circle is a square.
4. Having given a regular polygon of any number of sides inscribed in a circle, inscribe a regular polygon of double the number of sides.
5. If  $ABC$  an arc of a circle less than a semicircle be bisected in  $B$ , and  $AB$  produced meet  $CD$  which is drawn at right angles to  $BC$  in  $D$ , and the tangents at  $A$  and  $C$  meet in  $E$ , then  $B, C, D, E$  lie on a circle.

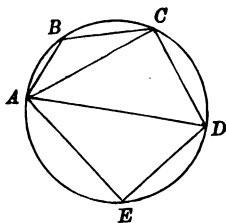
## PROPOSITION 31.

*An angle in a semicircle is a right angle ; an angle in an arc, which is greater than a semicircle, is less than a right angle ; and an angle in an arc, which is less than a semicircle, is greater than a right angle.*

Let  $ABCDE$  be a circle, of which  $ABD$  is a semicircle,  $ADC$  an arc greater than a semicircle, and  $ABC$  an arc less than a semicircle :

it is required to prove that the angle in the semicircle  $ABD$  is a right angle ; that the angle in the arc  $ADC$  is less than a right angle, and that the angle in the arc  $ABC$  is greater than a right angle.

CONSTRUCTION. Take any point  $B$  in the arc  $ABC$  and any point  $E$  in the semicircle  $AED$  and draw  $AB$ ,  $BC$ ,  $CD$ ,  $DE$ ,  $EA$ ,  $AC$ ,  $AD$ .



PROOF. Because  $ACDE$  is a quadrilateral inscribed in a circle, the sum of the angles  $ACD$ ,  $AED$  is equal to two right angles. (Prop. 22.)

But because each of the angles  $ACD$ ,  $AED$  is contained by a semicircle, the angle  $ACD$  is equal to the angle  $AED$  ;

(Prop. 27, Coroll.)

therefore each of them is a right angle.

Next, because the angle  $ACD$  of the triangle  $ACD$  is a right angle,

the angle  $ADC$  is less than a right angle,

(I. Prop. 17.)

and it is an angle in the arc  $ADC$ .

Again, because  $ABCD$  is a quadrilateral inscribed in a circle, the sum of the angles  $ABC$ ,  $ADC$  is equal to two right angles. (Prop. 22.)

And the angle  $ADC$  has been proved to be less than a right angle;

therefore the angle  $ABC$  is greater than a right angle,  
and it is an angle in the arc  $ABC$ .

Wherefore, *an angle in a semicircle &c.*

#### EXERCISES.

1. The circles described on two equal sides of a triangle as diameters intersect at the middle point of the third side.

2. The circles described on any two sides of a triangle as diameters intersect on the third side.

3. Construct the rectangle one of whose diagonals is a straight line given in magnitude and position and the other of whose diagonals passes through a given point.

4. An angle  $BAC$  of constant magnitude turns round its apex  $A$  which is fixed. Prove that the line joining the feet of the perpendiculars from a fixed point  $O$  on  $AB$ ,  $AC$  always touches a fixed circle.

5. If a circle  $A$  pass through the centre of a circle  $B$ , the tangents to  $B$  at the points of intersections of  $A$  and  $B$  intersect on the circle  $A$ .

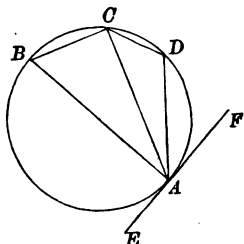
6. Two chords  $AB$ ,  $CD$  of constant length placed in a circle subtend angles at the centre whose sum is equal to two right angles. If  $AC$ ,  $BD$  intersect in  $P$ , the distances of  $P$  from the middle points of the chords will be independent of their relative positions.

## PROPOSITION 32.

*If a chord be drawn from the point of contact of a tangent to a circle, each of the angles which this chord makes with the tangent is equal to the angle in the alternate arc of the circle\*.*

Let  $ABCD$  be a circle, and  $EAF$  be the tangent at a point  $A$ , and let  $AC$  be a chord drawn from the point  $A$ : it is required to prove that the angle  $CAE$  is equal to the angle in the arc  $CDA$ , and the angle  $CAF$  to the angle in the arc  $CBA$ .

CONSTRUCTION. At  $A$  draw  $AB$  at right angles to  $EAF$ , cutting the circle again at  $B$ ; (I. Prop. 11.) take any point  $D$  in the arc  $ADC$  and draw  $BC$ ,  $CD$ ,  $DA$ .



PROOF. Because  $AB$  is drawn at right angles to  $EAF$ ,  $AB$  passes through the centre; (Prop. 19.) therefore  $BCDA$  is a semicircle; (Prop. 1 A.) and the angle  $BCA$  is a right angle; (Prop. 31.) therefore the sum of the other two angles  $BAC$ ,  $CBA$  of the triangle  $ABC$  is equal to a right angle. (I. Prop. 32.) And the sum of the angles  $BAC$ ,  $CAF$ , that is, the angle  $BAF$ , is a right angle; therefore the sum of the angles  $BAC$ ,  $CAF$  is equal to the sum of the angles  $BAC$ ,  $CBA$ .

\* The *alternate* arc is the name generally given to the arc which lies on the side of the chord opposite to the angle spoken of.



Take away the common angle  $BAC$ ;

then the angle  $CAF$  is equal to the angle  $CBA$ ,  
which is an angle in the arc  $ABC$ .

Again, because the sum of the angles  $ABC$ ,  $ADC$  is  
equal to two right angles, (Prop. 22.)

and the sum of the angles  $CAF$ ,  $CAE$  is equal to two  
right angles; (I. Prop. 13.)

therefore the sum of the angles  $CAF$ ,  $CAE$  is equal to  
the sum of the angles  $ABC$ ,  $ADC$ ;

and it has been proved that the angle  $CAF$  is equal  
to the angle  $ABC$ ;

therefore the angle  $CAE$  is equal to the angle  $ADC$ ,  
which is an angle in the arc  $ADC$ .

Wherefore, *if a chord be drawn &c.*

The theorem of Proposition 32 follows immediately from the theorem of Proposition 21, if we consider the tangent at a point as the limiting position of a chord drawn through the point. (See page 217.)

For the angle between the tangent  $AF$  in the diagram of Proposition 32 and the chord  $AC$  is the angle between two chords of the arc  $ABC$ , one an indefinitely short one drawn from a point indefinitely near  $A$  to  $A$  and the other drawn from the same point to  $C$ , and is therefore an angle in the arc  $ABC$  and therefore equal to the angle  $ABC$ . (Prop. 21.)

#### EXERCISES.

1. If two circles touch each other, any straight line drawn through the point of contact will cut off similar segments.

2. On the same side of portions  $AB$ ,  $AC$  of a straight line  $ABC$  similar segments of circles are described: prove that the circles touch one another.

3. If two circles  $OPQ$ ,  $Opq$  touch at  $O$ , and  $OPp$ ,  $OQq$  be straight lines, the chords  $PQ$ ,  $pq$  are parallel.

4. If a straight line cut two circles which touch at  $O$ , in the points  $P$ ,  $Q$ , and  $p$ ,  $q$ , the angles  $POp$ ,  $QOq$  are either equal or supplementary.

5.  $ABC$  is a triangle inscribed in a circle, and from any point  $D$  in  $BC$  a straight line  $DE$  is drawn parallel to  $CA$  and meeting the tangent at  $A$  in  $E$ ; shew that a circle may be described round  $AEBD$ .

## PROPOSITION 33.

*To describe on a given finite straight line an arc of a circle containing an angle equal to a given angle.*

Let  $AB$  be the given straight line, and  $C$  the given angle:

it is required to describe on  $AB$  an arc of a circle containing an angle equal to the angle  $C$ .

CONSTRUCTION. Bisect  $AB$  at  $E$ . (I. Prop. 10.)

If the angle  $C$  be a right angle, with  $E$  as centre and  $EA$  or  $EB$  as radius, describe a circle:

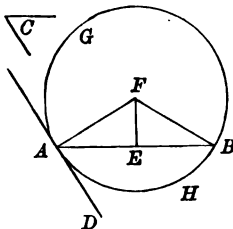
then the semicircle on either side of  $AB$  is an arc described as required. (Prop. 31.)

If the angle  $C$  be not a right angle, from  $A$  draw  $AD$  making the angle  $BAD$  equal to the angle  $C$ ; (I. Prop. 23.) and at  $A, E$  draw  $AF, EF$  at right angles to  $AD, AB$  respectively, (I. Prop. 11.)

and let them meet at  $F^*$ .

Draw  $FB$ , and with  $F$  as centre and  $FA$  as radius describe the circle  $AGH$ :

this circle passes through  $B$  and the arc  $AGB$  on the side of  $AB$  away from  $AD$  is an arc described as required.



PROOF. Because in the triangles  $FEA, FEB$ ,  
 $EA$  is equal to  $EB$ ,  
 and  $FE$  to  $FE$ ,  
 and the angle  $FEA$  to the angle  $FEB$ ,  
 the triangles are equal in all respects; (I. Prop. 4.)  
 therefore  $FA$  is equal to  $FB$ .

\* The lines must meet. See Ex. 2, p. 51.

Therefore the circle  $AGH$  passes through  $B$ .

Again, because  $AD$  is drawn from  $A$  at right angles to the radius  $AF$ ,

$AD$  touches the circle ; (Prop. 16.)

and because the chord  $AB$  is drawn from the point of contact of the tangent  $AD$ ,

the angle in the alternate arc  $AGB$  is equal to the angle  $DAB$ , (Prop. 32.)

that is, to the angle  $C$ .

Wherefore, *on the given straight line  $AB$  the arc  $AGB$  has been described containing an angle equal to the given angle  $C$ .*

### EXERCISES.

1. Find a point at which each of two given finite straight lines subtends a given angle.
2. Construct a triangle, having given the base, the vertical angle, and the foot of the perpendicular from the vertex on the base.
3. Having given the base and the vertical angle of a triangle, construct the triangle which will have the maximum area.
4. Find a point  $O$  within a given triangle  $ABC$ , so that, if  $AO$ ,  $BO$ ,  $CO$  be joined, the angles  $OAB$ ,  $OBC$ ,  $OCA$  shall be all equal.
5. Construct a triangle, having given the base, the vertical angle, and the altitude.
6. Find the locus of a point at which two given equal straight lines  $AB$ ,  $BC$  subtend equal angles.

## PROPOSITION 34.

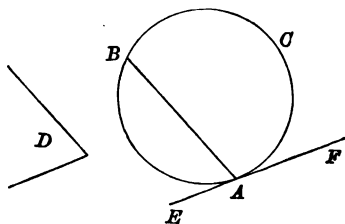
*To cut off from a given circle an arc containing an angle equal to a given angle.*

Let  $ABC$  be the given circle and  $D$  the given angle :  
it is required to cut off from the circle  $ABC$  an arc containing an angle equal to the angle  $D$ .

CONSTRUCTION. Take any point  $A$  on the circle, and through  $A$  draw the straight line  $EAF$  to touch the circle at  $A$ . (Prop. 17.)

From  $A$  draw  $AB$  making the angle  $BAE$  equal to the angle  $D$ ,

and cutting the circle again at  $B$ : (I. Prop. 23.)  
then the arc  $ACB$ , on the side of  $AB$  away from  $E$ , is an arc cut off as required.



PROOF. Because the chord  $AB$  is drawn from the point of contact  $A$  of the tangent  $EAF$ , the angle  $EAB$  is equal to the angle in the alternate arc  $ACB$ . (Prop. 32.)

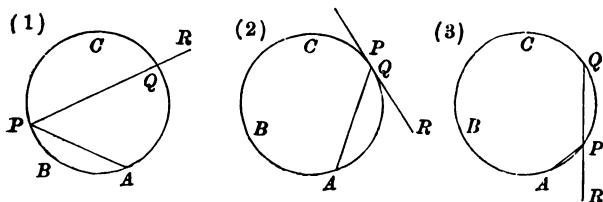
And the angle  $EAB$  is equal to the angle  $D$ ;  
therefore the angle in the arc  $ACB$  is equal to the angle  $D$ .

Wherefore, *the arc  $ACB$  has been cut off from the given circle  $ABC$  containing an angle equal to the given angle  $D$ .*

Through a given point *two* chords can be drawn which will cut off arcs containing an angle equal to a given angle.

*Outline of Alternative Construction.*

Through *A* draw any chord *AP*, and from *P* draw *PR* making the angle *RPA* equal to the given angle *D*, and cutting the circle again at *Q*.



It may be proved that the arc *ABQ*, measured from *A* on the side of *AP* opposite to that on which *PR* is drawn, is an arc cut off as required.

EXERCISES.

1. In a given circle inscribe an equiangular triangle.
2. Inscribe in a given circle a triangle, so that one angle may be a half of a second angle and a third of the third angle.
3. Inscribe in a given circle a right-angled triangle, so that one of its acute angles may be three times the other.

## PROPOSITION 35.

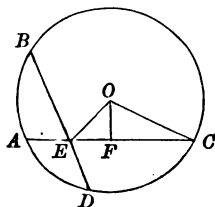
*If two chords of a circle intersect at a point within the circle, the rectangle contained by the segments of one chord is equal to the rectangle contained by the segments of the other chord.*

Let  $ABCD$  be a circle and  $AC$ ,  $BD$  two chords intersecting at the point  $E$  within the circle :

it is required to prove that the rectangle contained by  $AE$ ,  $EC$  is equal to the rectangle contained by  $BE$ ,  $ED$ .

CONSTRUCTION. If the point  $E$  be the centre, it is clear that  $AE$ ,  $EC$ ,  $BE$ ,  $ED$  are equal, each being a radius, and that the rectangles  $AE$ ,  $EC$  and  $BE$ ,  $ED$ , each of which is equal to the square on a radius, are equal.

If  $E$  be not the centre, find  $O$  the centre ; (Prop. 5.)  
draw  $OF$  at right angles to  $AC$ , (I. Prop. 12.)  
and draw  $OE$ ,  $OC$ .



PROOF. Because  $OF$  is drawn from the centre at right angles to the chord  $AC$ ,

therefore  $AF$  is equal to  $FC$ . (Prop. 4.)

And because  $EC$  is the sum of  $FC$ ,  $EF$ ,  
and  $AE$  is the difference of  $AF$ ,  $EF$ , that is, of  $FC$ ,  $EF$ ,  
therefore the rectangle  $AE$ ,  $EC$  is equal to the difference  
of the squares on  $FC$ ,  $EF$ . (II. Prop. 5.)

But because the angles at  $F$  are right angles,  
the sum of the squares on  $OF$ ,  $FC$  is equal to the square  
on  $OC$ ,  
and the sum of the squares on  $OF$ ,  $EF$  is equal to the square  
on  $OE$  ; (I. Prop. 47.)

therefore the difference of the squares on  $FC$ ,  $EF$  is equal to the difference of the squares on  $OC$ ,  $OE$ .

Therefore the rectangle  $AE$ ,  $EC$  is equal to the difference of the squares on  $OC$ ,  $OE$ .

Similarly it can be proved that the rectangle  $BE$ ,  $ED$  is equal to the difference of the squares on  $OB$ ,  $OE$ , that is, is equal to the difference of the squares on  $OC$ ,  $OE$ , since  $OC$  is equal to  $OB$ .

Therefore the rectangle  $AE$ ,  $EC$  is equal to the rectangle  $BE$ ,  $ED$ .

Wherefore, if two chords of a circle &c.

There are two special cases which should be noticed by the student, one case, when the points  $E$ ,  $F$  coincide, i.e. when one chord bisects the other; the other case, when the points  $O$ ,  $F$  coincide, i.e. when one chord is a diameter.

In Proposition 35 the distances between the ends of a chord and a point in the chord are spoken of as *segments of the chord*. In Proposition 36 it will be noticed that the expression *segments of a chord* has been used of the distances between the ends of the chord and a point taken in the chord produced. In the first case the chord is equal to the sum of the segments, in the second to the difference of the segments.

#### EXERCISES.

1. Prove the converse of Proposition 35, i.e. that, if  $AC$ ,  $BD$  be two straight lines intersecting at  $E$  such that the rectangles  $AE$ ,  $EC$ , and  $BE$ ,  $ED$  are equal, then  $A$ ,  $B$ ,  $C$ ,  $D$  lie on a circle.

2. If through any point in the common chord of two intersecting circles there be drawn any two other chords, one in each circle, their four extremities all lie on a circle.

3. Draw through a given point within a circle a chord, one of whose segments shall be four times as long as the other. When is this possible?

4. Divide a given straight line into two parts, so that the rectangle contained by the parts may be equal to a given rectangle.

5.  $A$ ,  $B$ ,  $C$  are three points on a circle,  $D$  is the middle point of  $BC$  and  $AD$  produced meets the circle in  $E$ : prove that the sum of the squares on  $AB$ ,  $AC$  is double of the rectangle  $AD$ ,  $AE$ .

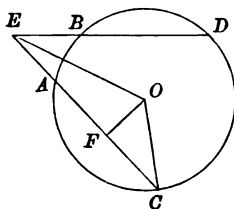
## PROPOSITION 36.

*If two chords of a circle when produced intersect at a point without the circle, the rectangle contained by the segments of one chord is equal to the rectangle contained by the segments of the other chord.*

Let  $ABDC$  be a circle and let  $CA, DB$  be two chords, which intersect, when produced beyond  $A$  and  $B$ , at the point  $E$  without the circle:

it is required to prove that the rectangle contained by  $EA, EC$  is equal to the rectangle contained by  $EB, ED$ .

CONSTRUCTION. Find  $O$  the centre; (Prop. 5.)  
draw  $OF$  at right angles to  $AC$ , (I. Prop. 12.)  
and draw  $OE, OC$ .



PROOF. Because  $OF$  is drawn from the centre at right angles to the chord  $AC$ ,

therefore  $AF$  is equal to  $FC$ . (Prop. 4.)

And because  $EC$  is the sum of  $EF, FC$ ,  
and  $EA$  is the difference of  $EF, AF$ , that is, of  $EF, FC$ ,  
therefore the rectangle  $EA, EC$  is equal to the difference  
of the squares on  $EF, FC$ . (II. Prop. 6.)

But because the angles at  $F$  are right angles,  
the sum of the squares on  $OF, FE$  is equal to the square  
on  $OE$ ,

and the sum of the squares on  $OF, FC$  is equal to the  
square on  $OC$ ; (I. Prop. 47.)

therefore the difference of the squares on  $EF, FC$  is equal  
to the difference of the squares on  $OE, OC$ .



Therefore the rectangle  $EA, EC$  is equal to the difference of the squares on  $OE, OC$ .

Similarly it can be proved that the rectangle  $EB, ED$  is equal to the difference of the squares on  $OE, OD$ , that is, is equal to the difference of the squares on  $OE, OC$ , since  $OD$  is equal to  $OC$ .

Therefore the rectangle  $EA, EC$  is equal to the rectangle  $EB, ED$ .

Wherefore, *if two chords of a circle &c.*

There are two special cases which should be noticed by the student, one case, when the points  $O, F$  coincide, i.e. when one chord is a diameter; the other case, when the points  $B, D$  coincide, i.e. when one chord is a tangent. The statement of the theorem in the latter case appears in the Corollary.

#### COROLLARY.

*If a chord of a circle be produced to any point, the rectangle contained by the segments of the chord is equal to the square on the tangent drawn to the circle from the point.*

This result is seen at once on considering the tangent as the limiting position of the secant.

#### EXERCISES.

1. Prove the converse of Proposition 36, i.e. that, if  $EAC, EBD$  be two straight lines intersecting at  $E$  such that the rectangles  $EA, EC$  and  $EB, ED$  are equal, then  $A, B, C, D$  lie on a circle.

2. If two circles intersect each other, their common chord bisects their common tangents.

3. From a given point as centre describe a circle cutting a given straight line in two points, so that the rectangle contained by their distances from a given point in the straight line may be equal to a given square.

4. If  $ABC$  be a triangle and  $D$  a point in  $AC$  such that the angle  $ABD$  is equal to the angle  $ACB$ , then the rectangle  $AC, AD$  is equal to the square on  $AB$ .

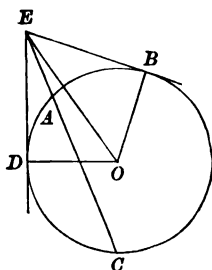
5. If from each of two given points, a pair of tangents be drawn to a given circle, the middle points of the chords joining the points of contact of each pair of tangents lie on the circumference of a circle passing through the two given points.

## PROPOSITION 37.

*If from an external point there be drawn to a circle two straight lines, one of which cuts the circle in two points and the other meets it, and if the rectangle contained by the segments of the chord on the line which cuts the circle be equal to the square on the line which meets the circle, the line which meets the circle is a tangent to it.*

Let  $ABCD$  be a circle and  $E$  an external point; and let  $EAC$  be a straight line cutting the circle at  $A$ ,  $C$  and  $EB$  a straight line meeting it at  $B$ , such that the rectangle contained by  $EA$ ,  $EC$  is equal to the square on  $EB$ : it is required to prove that  $EB$  touches the circle.

CONSTRUCTION. Find the centre  $O$ ; (Prop. 5.)  
from  $E$  draw  $ED$  to touch the circle at  $D$ ; (Prop. 17.)  
and draw  $OB$ ,  $OD$ ,  $OE$ .



PROOF. Because  $ED$  is a tangent, and  $OD$  is the radius, the angle  $EDO$  is a right angle. (Prop. 18.)

Because  $EAC$  cuts the circle and  $ED$  touches it, the rectangle  $EA$ ,  $EC$  is equal to the square on  $ED$ ; (Prop. 36, Coroll.)

and the rectangle  $EA$ ,  $EC$  is equal to the square on  $EB$ ; therefore the square on  $EB$  is equal to the square on  $ED$ ; therefore  $EB$  is equal to  $ED$ .

Again, because in the triangles  $EBO$ ,  $EDO$ ,

$EB$  is equal to  $ED$ ,

and  $OB$  to  $OD$ ,

and  $OE$  to  $OE$ ,

the triangles are equal in all respects; (I. Prop. 8.)  
therefore the angle  $EBO$  is equal to the angle  $EDO$ .

But  $EDO$  is a right angle;

therefore the angle  $EBO$  is a right angle.

And because  $BE$  is at right angles to the radius  $OB$ ,

$BE$  touches the circle. (Prop. 16.)

Wherefore, *if from an external point &c.*

### EXERCISES.

1. If three circles meet two and two, the common chords of each pair meet in a point.

2. If three circles touch two and two, the tangents at the points of contact meet at a point.

3. If the tangents drawn to two intersecting circles from a point be equal, the common chord of the circles passes through the point.

4. Describe a circle which shall touch a given straight line at a given point, and shall cut off from another given straight line a chord of a given length.

5. On  $OP$ , the straight line drawn from a given point  $O$  to  $P$  a point on a given straight line, a point  $Q$  is taken such that the rectangle  $OP$ ,  $OQ$  is constant: prove that the locus of  $Q$  is a circle.

6. On  $OP$ , a chord of a given circle drawn from a given point  $O$ , a point  $Q$  is taken such that the rectangle  $OP$ ,  $OQ$  is constant: prove that the locus of  $Q$  is a straight line.

## PROPOSITION 37 A.

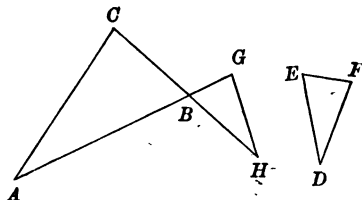
If two triangles be equiangular to one another, the rectangle contained by any side of the one and any side of the other is equal to the rectangle contained by the corresponding sides\*.

Let  $ABC$ ,  $DEF$  be two triangles which are equiangular to one another, having the angles at  $A$ ,  $B$ ,  $C$  equal to the angles at  $D$ ,  $E$ ,  $F$  respectively :

it is required to prove that the rectangle  $AB$ ,  $EF$  is equal to the rectangle  $BC$ ,  $DE$ .

CONSTRUCTION. In  $AB$ ,  $CB$  produced beyond  $B$ , take points  $G$ ,  $H$  such that  $BG$  is equal to  $EF$ , and  $BH$  to  $ED$ ;  
(I. Prop. 3.)

and draw  $GH$ .



PROOF. Because the angle  $GBH$  is equal to the angle  $CBA$ ,  
(I. Prop. 15.)

and the angle  $FED$  is equal to the angle  $CBA$ ,

the angle  $GBH$  is equal to the angle  $FED$ .

Because in the triangles  $BGH$ ,  $EFD$ ,

$BG$  is equal to  $EF$  and  $BH$  to  $ED$ ,

and the angle  $GBH$  is equal to the angle  $FED$ ,

the triangles are equal in all respects: (I. Prop. 4.)

therefore the angle  $BGH$  is equal to the angle  $EFD$ ;

but the angle  $EFD$  is equal to the angle  $BCA$ ,

therefore the angle  $AGH$  is equal to the angle  $ACH$ ;

therefore the points  $A$ ,  $C$ ,  $G$ ,  $H$  lie on a circle.

(Prop. 21, Coroll.)

Therefore the rectangle  $AB$ ,  $BG$  is equal to the rectangle

$CB$ ,  $BH$ ; (Prop. 35.)

that is, the rectangle  $AB$ ,  $EF$  is equal to the rectangle  $BC$ ,  $DE$ .

Wherefore, if two triangles &c.

\* In two triangles which are equiangular to one another, two sides are said to correspond when they are opposite to equal angles.

## PROPOSITION 37 B.

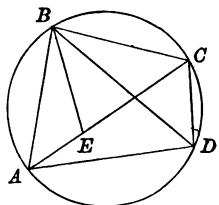
*The rectangle contained by the diagonals of a convex quadrilateral inscribed in a circle is equal to the sum of the rectangles contained by pairs of opposite sides\*.*

Let  $ABCD$  be a quadrilateral inscribed in a circle and  $AC$ ,  $BD$  be its diagonals:

it is required to prove that the rectangle  $AC$ ,  $BD$  is equal to the sum of the rectangles  $AB$ ,  $CD$  and  $BC$ ,  $AD$ .

CONSTRUCTION. From  $B$  in  $BA$ , on the same side of  $BA$  as  $CD$ , draw  $BE$  making the angle  $ABE$  equal to the angle  $CBD$ , and meeting  $AC$  in  $E$ . (I. Prop. 23.)

PROOF. Because the angle  $BAC$  is equal to the angle  $BDC$ , (Prop. 21.) and the angle  $ABE$  is equal to the angle  $DBC$ , (Constr.) therefore the triangles  $ABE$ ,  $DBC$  are equiangular to one another; (I. Prop. 32.)



therefore the rectangle  $AB$ ,  $CD$  is equal to the rectangle  $AE$ ,  $BD$ . (Prop. 37 A.)

Again, because the angle  $ABE$  is equal to the angle  $DBC$ , the angle  $ABD$  is equal to the angle  $EBC$ ; and the angle  $BDA$  is equal to the angle  $BCA$  (i.e.  $BCE$ ), (Prop. 21.)

therefore the triangles  $ABD$ ,  $EBC$  are equiangular to one another; (I. Prop. 32.)

therefore the rectangle  $AD$ ,  $BC$  is equal to the rectangle  $EC$ ,  $BD$ ;

but it has been proved that

the rectangle  $AB$ ,  $CD$  is equal to the rectangle  $AE$ ,  $BD$ .

Therefore the sum of the rectangles  $AB$ ,  $CD$  and  $AD$ ,  $BC$  is equal to the sum of the rectangles  $AE$ ,  $BD$  and  $EC$ ,  $BD$ , that is, to the rectangle  $AC$ ,  $BD$ . (II. Prop. 1.)

Wherefore, *the rectangle contained &c.*

\* This theorem is attributed to Ptolemy, a Greek geometer of Alexandria, who died about A.D. 160.

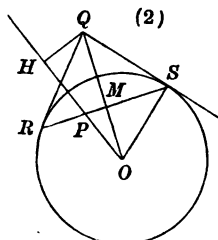
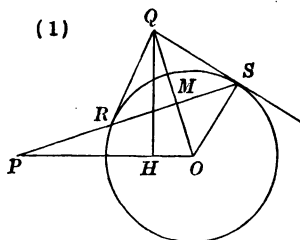
## ADDITIONAL PROPOSITION.

If a straight line be drawn through a given point to cut a given circle, the intersection of the tangents at the two points of section always lies on a fixed straight line\*.

Let  $PRS$  be any straight line drawn through a given point  $P$  to cut a given circle, whose centre is  $O$ , in  $R$  and  $S$ .

Let  $QR$ ,  $QS$  be the tangents at  $R$ ,  $S$ .

Draw  $OQ$  intersecting  $RS$  in  $M$ ; and draw  $OP$ , and draw  $QH$  perpendicular to  $OP$  or  $OP$  produced.



Because the angle at  $M$  is a right angle (Ex. 3, page 217),

and the angle at  $H$  is a right angle,

the points  $P$ ,  $H$ ,  $M$ ,  $Q$  lie on a circle;

fig. 1 (Prop. 21, Coroll.) and fig. 2 (Prop. 22, Coroll.)

therefore the rectangle  $OH$ ,  $OP$  is equal to the rectangle  $OM$ ,  $OQ$ .

(Prop. 36.)

But because  $QS$  is a tangent at  $S$ ,

the angle  $OSQ$  is a right angle, (Prop. 18.)

and the angle at  $M$  is a right angle,

therefore the rectangle  $OM$ ,  $OQ$  is equal to the square on  $OS$ ;

(I. Prop. 47.)

Therefore the rectangle  $OP$ ,  $OH$  is equal to the square on  $OS$ .

But  $OP$  and  $OS$  are both constants,

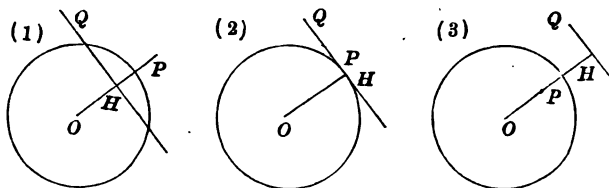
therefore  $OH$  is a constant,

and the point  $Q$  always lies on a fixed straight line,

i.e. the line drawn through the fixed point  $H$  at right angles to  $OP$ .

\* This line is called the **polar** of the given point, and the point is called the **pole** of the line with respect to the circle.

It has now been proved that, if a point and a straight line be such that the straight line joining the centre of a circle to the point is at right angles to the line, and the rectangle contained by the distances of the point and the line from the centre is equal to the square on the radius of the circle, the point is the pole of the line, and the line the polar of the point with respect to the circle.



In the diagram  $O$  is the centre of the circle:  $H$  is a point in  $OP$  such that the rectangle  $OP, OH$  is equal to the square on the radius, and  $HQ$  is at right angles to  $OP$ .

$P$  is the pole of  $HQ$ , and  $HQ$  is the polar of  $P$ .

It will be observed that,

if  $P$  be without the circle (fig. 1), the polar cuts the circle:

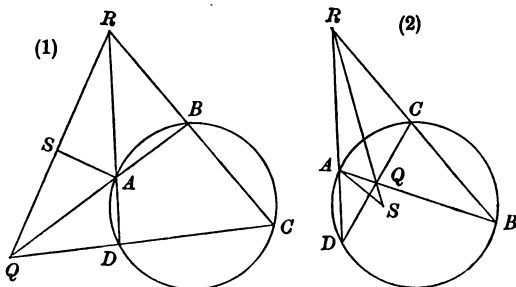
if  $P$  be on the circle (fig. 2), the polar is the tangent to the circle, and if  $P$  be within the circle (fig. 3), the polar does not cut the circle.

## ADDITIONAL PROPOSITION.

*If a quadrilateral be inscribed in a circle, the square on the straight line joining the points of intersection of opposite sides is less than the sum of the squares on the straight lines joining those points to the centre of the circle by twice the square on the radius of the circle.*

Let  $ABCD$  be a quadrilateral inscribed in a circle whose centre is  $O$ ; and let the sides  $AB, CD$  meet in  $Q$  and the sides  $AD, BC$  in  $R$ .

Draw  $QR$  and draw  $AS$  making the angle  $RAS$  equal to the angle  $RQD$  and meeting  $RQ$  in  $S$ . (I. Prop. 23.)



Because in the triangles  $RAS, RQD$ ,  
the angle  $RAS$  is equal to the angle  $RQD$ ,  
the angle  $RSA$  is equal to the angle  $RDQ$ ; (I. Prop. 32.)  
therefore the points  $S, A, D, Q$  lie on a circle.

(Prop. 21 or 20, Coroll.)

Therefore the rectangle  $RS, RQ$  is equal to the rectangle  $RA, RD$ .  
(Prop. 36.)

Also it can be proved that the points  $R, S, A, B$  lie on a circle.  
Therefore the rectangle  $QS, QR$  is equal to the rectangle  $QA, QB$ .  
(Prop. 35 or 36.)

Therefore the square on  $QR$ ,  
in figure (1), being the sum of the rectangles  $RS, RQ$  and  $QS, QR$ ,  
is equal to the sum of the rectangles  $RA, RD$  and  $QA, QB$ ;  
and in figure (2), being the difference of the rectangles  $RS, RQ$ , and  
 $QS, QR$ , is equal to the difference of the rectangles  $RA, RD$   
and  $QA, QB$ .

Since the rectangle  $RA, RD$  is equal to the difference of the squares on  
 $RO$  and the radius,  
(Prop. 36.)



and the rectangle  $QA, QB$  in figure (1) is equal to the difference of the squares on  $QO$  and the radius, (Prop. 36.)

and in figure (2) is equal to the difference of the squares on the radius and  $QO$ ; (Prop. 35.)

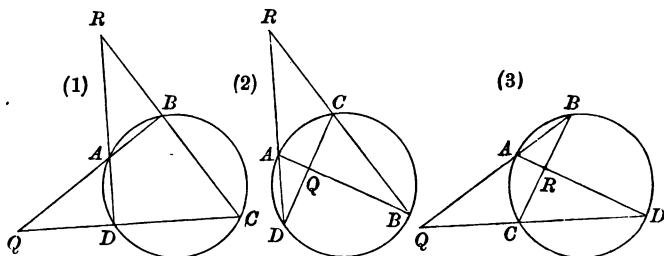
it follows that in both cases

the square on  $QR$  is less than the sum of the squares on  $QO, RO$  by twice the square on the radius.

### ADDITIONAL PROPOSITION.

*If one pair of opposite sides of a quadrilateral inscribed in a circle intersect at a fixed point, the other pair of opposite sides intersect on a fixed straight line\*.*

Let  $ABCD$  be a quadrilateral inscribed in a circle, whose centre is  $O$ ; and let the sides  $AB, CD$  meet in  $Q$  and  $AD, BC$  in  $R$ .



Because  $Q$  and  $R$  are the intersections of opposite sides of a quadrilateral inscribed in the circle,

the square on  $QR$  is less than the squares on  $OQ, OR$  by twice the square on the radius; (Add. Prop. page 260.)

therefore the difference of the squares on  $QR, OQ$  is equal to the difference of the square on  $OR$  and twice the square on the radius, which is a constant, if the point  $R$  be fixed.

Therefore the locus of the point  $Q$  is a straight line.

(Ex. 2, page 125.)

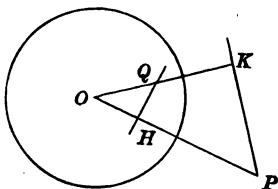
\* We leave to the student as an exercise the proof that this line is the polar of the fixed point.

## ADDITIONAL PROPOSITION.

*If one point lie on the polar of another point, the second point lies on the polar of the first point.*

Let  $P, Q$  be two points such that  $Q$  lies on the polar of  $P$ ,  
*i.e.* if  $QH$  be drawn perpendicular to  $OP$ ,  
 the rectangle  $OH, OP$  is equal to the square on the radius.

CONSTRUCTION. Draw  $PK$  perpendicular to  $OQ$ . (I. Prop. 12.)



PROOF. Because the angles at  $H$  and  $K$  are right angles,  
 $Q, K, P, H$  lie on a circle; (Prop. 22, Coroll.)  
 therefore the rectangle  $OQ, OK$  is equal to the rectangle  $OH, OP$ ,  
 (Prop. 36.)

and therefore to the square on the radius;  
 and  $KP$  is at right angles to  $OK$ ;  
 therefore  $KP$  is the polar of  $Q$ ,  
 or, in other words,  $P$  lies on the polar of  $Q$ .

## EXERCISES.

1. Prove that the polar of a point without a circle is the straight line joining the points of contact of tangents drawn from the point to the circle.

2. If  $O$  be the centre of a circle, and the polar of a point  $P$  cut  $PO$  in  $H$ , and any straight line through  $P$  cut the circle in  $R$  and  $S$ , then the polar bisects the angle  $RHS$ .

3. If a straight line  $PQR$  cut a circle in  $Q$  and  $R$  and cut the polar of  $P$  in  $K$ , and if  $M$  be the middle point of  $QR$ , then the rectangles  $PQ, PR$  and  $PK, PM$  are equal.

4. If  $P, Q, R, S$  be the points of contact of the sides  $AB, BC, CD, DA$  of a quadrilateral  $ABCD$  with an inscribed circle, the straight lines  $AC, BD, PR, QS$  are concurrent.

5. Shew how to draw two tangents to a given circle from a given external point by means of straight lines only.

6. Shew how to draw a tangent to a given circle at a given point on it by means of straight lines only.

## ADDITIONAL PROPOSITION.

*The locus of a point from which tangents drawn to two given circles are equal is a straight line\*.*

Let  $P$  be a point such that  $PQ$ ,  $PR$  tangents drawn to two given circles are equal.

Find the centres  $A$ ,  $B$  of the circles; (Prop. 5.)  
draw  $AB$ ,  $AP$ ,  $AQ$ ,  $BP$ ,  $BR$ , and  
draw  $PH$  perpendicular to  $AB$ .

(I. Prop. 12.)

Because  $PQ$ ,  $PR$  are tangents

the angles at  $Q$  and  $R$  are right angles.

Therefore the sum of the squares on  $PQ$ ,  $AQ$  is equal to the square on  $AP$ , (I. Prop. 47.)

and the sum of the squares on  $PR$ ,  $BR$  is equal to the square on  $BP$ ;  
therefore the difference of the squares on  $AQ$ ,  $BR$  is equal to the  
difference of the squares on  $AP$ ,  $BP$ .

But because the angles at  $H$  are right angles,

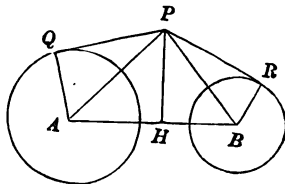
the difference of the squares on  $AP$ ,  $BP$

is equal to the difference of the squares on  $AH$ ,  $HB$ .

Therefore the difference of the squares on  $AH$ ,  $HB$  is equal to the  
difference of the squares on  $AQ$ ,  $BR$ , which is a constant;

therefore  $H$  is a fixed point,

and the straight line  $HP$  on which  $P$  lies is drawn through  $H$  at  
right angles to  $AB$  the line of the centres, and is therefore a fixed  
straight line.



\* This line is called the **Radical Axis** of the two circles. This name was given to the line by L. Gaultier de Tours, a French geometer. See *Journal de l'école Polytechnique*, tom. ix. p. 139 (1813).

EXERCISES.

1. Prove that the radical axis of two intersecting circles passes through their points of intersection.
2. What is the radical axis of two circles which touch each other?
3. Prove that the middle points of the four common tangents of two circles external to each other lie on a straight line.
4. Prove that the radical axes of three circles taken two and two together meet in a point\*.
5. Shew how to draw the radical axis of two circles which do not meet.
6. Draw a circle passing through a given point and cutting two given circles so that its chords of intersection with the two circles may each pass through given points.
7.  $O$  is a fixed point outside a given circle: find a straight line such that each of the tangents drawn from any point  $P$  in that line to the circle shall be equal to  $PO$ .
8. Draw a straight line in a given direction so that chords cut from it by two given circles may be equal.
9. Prove that the difference of the squares of the tangents from any point to two circles is equal to twice the rectangle under the distance between their centres and the distance of the point from their radical axis.
10. Through two given points draw a circle to cut a given circle in such a way that the angle contained in the segment cut off the given circle may be equal to a given angle.

\* This point is called the **Radical Centre** of the three circles.

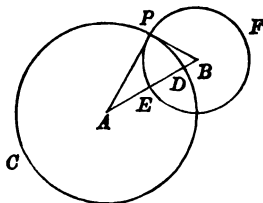
**DEFINITION.** *Two circles or other curves, which meet at a point, are said to meet at the angle at which their tangents at the point meet.*

*Two circles or other curves are said to be orthogonal or to cut orthogonally at a point, when they intersect at right angles at the point.*

#### ADDITIONAL PROPOSITION.

*If the square on the distance between the centres of two circles be equal to the sum of the squares on the radii, the circles are orthogonal.*

Let  $A, B$  be the centres of two circles  $CPD, EPF$ , which intersect at  $P$ , and are such that the square on  $AB$  is equal to the sum of the squares on  $AP, BP$ .



Because the square on  $AB$  is equal to the sum of the squares on  $AP, BP$ ,

the angle  $APB$  is a right angle. (I. Prop. 48.)

And  $BP$  is a radius of the circle  $EPF$ ;

therefore  $AP$  touches the circle  $EPF$ .

Similarly it can be proved that  $BP$  touches the circle  $CPD$ ;

therefore the circles  $CPD, EPF$  are orthogonal.

**COROLLARY.** *The radius of each of two orthogonal circles drawn to a point of intersection is a tangent to the other circle.*

EXERCISES.

1. A circle, which passes through a given point and cuts a given circle orthogonally, passes through a second fixed point.
2. Describe a circle to cut a given circle orthogonally at two given points.
3. Describe a circle through two given points to cut a given circle orthogonally.
4. Two chords  $AD$ ,  $BC$  of a circle  $ACDB$ , of which  $AB$  is a diameter, intersect at  $E$ : a circle described round  $CDE$  will cut the circle  $ACDB$  at right angles.
5. Two circles cut each other at right angles in  $A$ ,  $B$ ;  $P$  is any point on one of the circles, and the lines  $PA$ ,  $PB$  cut the other circle in  $Q$ ,  $R$ : shew that  $QR$  is a diameter.
6. The internal and external bisectors of the vertical angle  $A$  of the triangle  $ABC$  meet the base in  $D$  and  $E$  respectively. Prove that the circles described about the triangles  $ABD$  and  $ABE$  cut at right angles, as also do those described about the triangles  $ACD$  and  $ACE$ .

## ADDITIONAL PROPOSITION.

*Every circle, which cuts two given circles orthogonally, has its centre on the radical axis of the given circles, and if it cut the straight line joining their centres, it cuts it in two fixed points.*

Let  $A, B$  be the centres of two given circles and let  $P$  be the centre of a circle, which cuts the given circles orthogonally at  $Q$  and  $R$ .

Draw  $AB, PQ, PR$ , and draw  $PH$  perpendicular to  $AB$ .

Because the circles cut orthogonally at  $Q$ ,

$PQ$  is a tangent at  $Q$ .

(Add. Prop. page 266, Coroll.)

Similarly it can be proved that

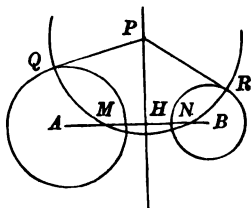
$PR$  is a tangent at  $R$ .

But  $PQ$  is equal to  $PR$ ;

therefore  $P$  is a point on the radical

axis of the given circles,

and therefore  $H$  is a fixed point.



(Add. Prop. page 264.)

Next, let the circle whose centre is  $P$  cut the line  $AB$  in  $M, N$ .

Because the circles cut orthogonally at  $Q$ ,

$AQ$  is a tangent to the circle  $QMNR$  at  $Q$ ,

and therefore the square on  $AQ$  is equal to the rectangle  $AM, AN$ ;

but because  $PH$  is at right angles to  $MN$ ,

$MH$  is equal to  $NH$ ; (Prop. 4.)

and the rectangle  $AM, AN$  is equal to the difference of the squares on  $AH, MH$ ; (II. Prop. 6.)

therefore the square on  $AQ$  is equal to the difference of the squares on  $AH, MH$ .

Now the lines  $AQ, AH$  are of constant length;

therefore  $MH$  (or  $NH$ ) is of constant length.

Therefore the points  $M$  and  $N$  are at a constant distance from  $H$ , which is a fixed point;

therefore the points  $M$  and  $N$  are fixed points.



We leave it to the student as an exercise to prove that:

- if the given circles be external to each other, the points  $M$  and  $N$  are real, one within each of the given circles;
- if the circles touch externally,  $M$  and  $N$  coincide with the point of contact;
- if the circles intersect, the circle, whose centre is  $P$ , does not intersect the line  $AB$  in real points;
- if one circle touch the other internally, the points are again real, and they coincide with the point of contact;
- if one circle lie wholly within the other, the points  $M$  and  $N$  are both real, one within both circles and the other without both circles.

### EXERCISES.

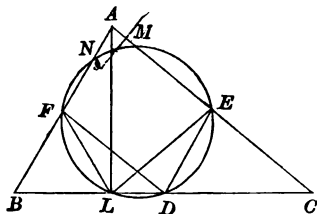
1. Draw a circle to cut three given circles orthogonally.
2. Prove that every pair of circles, which cut two given circles orthogonally, has the same radical axis.
3. Of four given circles three have their centres in the same straight line, and the fourth cuts the other three orthogonally; prove that the radical axis of each pair of the three circles is the same.
4.  $ABCD$  is a quadrilateral inscribed in a circle; the opposite sides  $AB$  and  $DC$  are produced to meet at  $F$ ; and the opposite sides  $BC$  and  $AD$  at  $E$ : shew that the circle described on  $EF$  as diameter cuts the circle  $ABCD$  at right angles.
5. Find a point such that its polar with respect to each of two given circles is the same.

## ADDITIONAL PROPOSITION.

*The middle points of the sides of a triangle and the feet of the perpendiculars from the angular points on the opposite sides lie on a circle.*

Let  $D, E, F$  be the middle points of the sides  $BC, CA, AB$  of a triangle  $ABC$ , and  $L, M, N$  the feet of the perpendiculars on them from  $A, B, C$ .

Draw  $FL, LE, FD, DE$ .



Then because  $ALB$  is a right-angled triangle, and  $F$  is the middle point of  $AB$ ,

$FL$  is equal to  $FA$ ; (Ex. 7, page 87.)

therefore the angle  $FLA$  is equal to the angle  $FAL$ . (I. Prop. 5.)

Similarly it can be proved that the angle  $ALE$  is equal to the angle  $LAE$ ;

therefore the angle  $FLE$  is equal to the angle  $BAC$ .

Again because  $FD$  is parallel to  $AC$   
and  $DE$  to  $BA$ , (Add. Prop. page 101.)

$FAED$  is a parallelogram,

and the angle  $FDE$  is equal to the angle  $BAC$ ; (I. Prop. 34.)

therefore the angle  $FLE$  is equal to the angle  $FDE$ .

Therefore  $L, D, E, F$  lie on a circle. (Prop. 21, Coroll.)

Similarly it can be proved that  $M, D, E, F$  lie on a circle,

and that  $N, D, E, F$  lie on a circle.

But only one circle can be described through the three points  $D, E, F$ ;  
therefore these three circles are coincident.

Therefore the six points  $L, M, N, D, E, F$  lie on a circle.

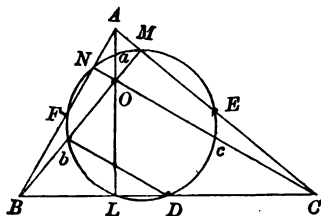
## ADDITIONAL PROPOSITION.

*The circle through the middle points of the sides of a triangle passes through the middle points of the straight lines joining the angular points of the triangle to the orthocentre.*

Let  $D, E, F$  be the middle points of the sides  $BC, CA, AB$  of a triangle  $ABC$ .

Draw  $AL, BM, CN$  perpendicular to  $BC, CA, AB$ , intersecting at  $O$ . (Add. Prop. page 95.)

Bisect  $AO, BO, CO$  at  $a, b, c$ .



In the triangle  $OBC$ ,  $D, c, b$  are the middle points of the sides, and  $L, M, N$  are the feet of the perpendiculars from the vertices on the opposite sides;

therefore  $L, D, c, M, N, b$  lie on a circle. (Add. Prop. page 270.)

Similarly it can be proved that

$L, c, E, M, a, N$  lie on a circle,

and that  $L, M, a, N, F, b$  lie on a circle.

But only one circle can be described through the three points  $L, M, N$ ;

therefore these circles are coincident.

Therefore the nine points  $L, D, c, E, M, a, N, F, b$  lie on a circle\*.

\* This circle is called the **Nine Point Circle** of the triangle.

## ADDITIONAL PROPOSITION.

*The feet of the perpendiculars drawn from any point on a circle to the three sides of a triangle inscribed in the circle lie on a straight line\*.*

Let  $ABC$  be a triangle and  $PL, PM, PN$  be the perpendiculars from a point  $P$  on the circle  $ABC$  to  $BC, CA, AB$ .

Draw  $LN, NM$ .

Because  $PLB, PNB$  are right angles,  
a circle can be described about  $PNLB$ ;

(III. Prop. 21, Coroll.)

therefore the angles  $PNL, PBL$  are supplementary. (III. Prop. 22.)

But the angles  $PAC, PBC$  are supplementary; (III. Prop. 22.)  
therefore the angle  $PNL$  is equal to the angle  $PAC$ .

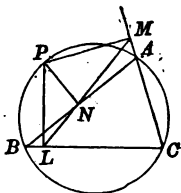
Again because  $PMA, PNA$  are right angles,

a circle can be described about  $PNAM$ ; (III. Prop. 22, Coroll.)  
therefore the angle  $PNM$  is equal to the angle  $PAM$ .

Therefore the sum of the angles  $PNL, PNM$  is equal to the sum of the angles  $PAC, PAM$ ,

that is, to two right angles. (I. Prop. 13.)

Therefore  $LN, NM$  are in the same straight line. (I. Prop. 14.)



## EXERCISES.

1. If  $PL, PM, PN$  be the perpendiculars drawn from  $P$  a point on the circle  $ABC$  to the sides  $BC, CA, AB$  of an inscribed triangle, and straight lines  $Pl, Pm, Pn$  be drawn to meet the sides in  $l, m, n$  such that the angles  $LPl, MPm, NPn$  are equal and measured in the same sense, then  $l, m, n$  are collinear.

2.  $P$  is a point on the circle circumscribing the triangle  $ABC$ . The pedal line of  $P$  cuts  $AC$  and  $BC$  in  $M$  and  $L$ .  $Y$  is the foot of the perpendicular from  $P$  on the pedal line. Prove that the rectangles  $PY, PC$ , and  $PL, PM$  are equal.

\* This line is called the **Pedal Line**.

Its discovery is attributed to Dr Robert Simson, and it is in consequence also called **Simson's Line**.

## MISCELLANEOUS EXERCISES.

1. If any point  $P$  on a fixed circle be joined to a fixed point  $A$ , and  $AQ$  be drawn equal to  $AP$  at a constant inclination  $PAQ$  to  $AP$ , the locus of  $Q$  is an equal circle.

2. Draw a straight line from one circle to another, to be equal and parallel to a given straight line.

3. Construct a parallelogram, two of whose angular points are given and the other two lie on two given circles.

Discuss the number of possible solutions of the problem in the different cases which may occur.

4. Find the locus of the centre of a circle whose circumference passes through two given points.

5. Shew that the straight lines drawn at right angles to the sides of a rectilineal figure inscribed in a circle from their middle points intersect at a fixed point.

6. Determine the centre of a given circle by means of a ruler with parallel edges whose breadth is less than the diameter of the circle.

7. A circle is described on the radius of another circle as diameter. Prove that any chord of the greatest circle drawn from the point of contact is bisected by the lesser circle.

8. Two circles  $DFC$ ,  $GEC$  whose centres are  $A$  and  $B$  intersect at  $C$ ; through  $C$ ,  $DCE$  and  $FCG$  are drawn equally inclined to  $AB$ : shew that  $DE$  and  $FG$  are equal.

9.  $AB$  and  $CD$  are two chords of a circle cutting at a point  $E$  within the circle;  $AB$  is produced to  $H$  so that  $BH$  is equal to  $BE$ . The circles  $AEC$  and  $ACH$  cut  $BC$  in  $K$  and  $L$ ; prove that  $B$  is the middle point of  $KL$ .

10. Through either of the points of intersection of two given circles draw the greatest possible straight line terminated both ways by the two circumferences.

11. Through two points  $A$ ,  $B$  on the same diameter of a circle and equidistant from its centre two parallel straight lines  $AP$ ,  $BQ$  are drawn towards the same parts, meeting the circle in  $P$  and  $Q$ : shew that  $PQ$  is perpendicular to  $AP$  and  $BQ$ .

12.  $A$  is a fixed point in the circumference of a circle and  $ABC$  an inscribed triangle such that the sum of the squares on  $AB, AC$  is constant; shew that the locus of the middle point of  $BC$  is a straight line.

13. From one of the points of intersection of two circles straight lines are drawn equally inclined to the common chord of the circles: prove that the portions of these lines, intercepted between the other points in which they meet the circumferences of the circles, are equal.

14. Describe a circle of given radius to touch two given circles.

Discuss the number of possible solutions.

15. Describe three circles to have their centres at three given points and to touch each other in pairs.

How many solutions are there?

16. Describe a circle to touch two given concentric circles and to pass through a given point.

Discuss the number of solutions. Is a solution always possible?

17. If  $AB, CD$  be two equal chords of a circle, one of the pairs of lines  $AD, BC$  and  $AC, BD$  are equal, and the other pair parallel.

18. If two equal chords  $AB, CD$  of a circle intersect at  $E$ ,  $AE$  is equal to  $CE$  or  $DE$ .

19. A quadrilateral is described about a circle: shew that two of its sides are together equal to the other two sides.

20. Shew that every parallelogram described about a circle is a rhombus.

21. If the tangents at two points where a straight line meets two circles, one point being on each circle, be parallel, the other pair also are parallel.

22. Two straight lines  $ABD, ACE$  touch a circle at  $B$  and  $C$ ; if  $DE$  be joined,  $DE$  is equal to  $BD$  and  $CE$  together: shew that  $DE$  touches the circle.

23. When an equilateral polygon of an even number of sides is described about a circle, the alternate angles are equal.

24. If a quadrilateral be described about a circle, the angles subtended at the centre of the circle by any two opposite sides of the figure are together equal to two right angles.

25. Two radii of a circle at right angles to each other when produced are cut by a straight line which touches the circle: shew that the tangents drawn from the points of section are parallel to each other.

26. If the straight line joining the centres of two circles, which are external to one another, cut them in the points  $A, B, C, D$ , the squares on the common tangents to the two circles are equal to the rectangles  $BD, AC$ , and  $BC, AD$ .

What is the corresponding theorem for two intersecting circles?

27.  $APB$  is an arc of a circle less than a semicircle; tangents are drawn at  $A$  and  $B$  and at any intermediate point  $P$ ; shew that the sum of the sides of the triangle formed by the three tangents is invariable for all positions of  $P$ .

28. Given a circle, and a straight line in its plane; draw a tangent to the circle, which shall make a given angle with the given straight line.

29. Find the point in the circumference of a given circle, the sum of whose distances from two given straight lines at right angles to each other, which do not cut the circle, is the greatest or least possible.

30. A straight line is drawn touching two circles: shew that the chords are parallel which join the points of contact and the points where the straight line through the centres meets the circumferences.

31. From the centre  $C$  of a circle,  $CA$  is drawn perpendicular to a given straight line  $AB$ , which does not meet the circle, and in  $AC$  a point  $P$  is taken, such that  $AP$  is equal to the length of the tangent from  $A$ : prove that, if  $Q$  be any point in  $AB$ ,  $QP$  is equal to the length of the tangent from  $Q$ .

32. If a quadrilateral, having two of its sides parallel, be described about a circle, a straight line drawn through the centre of the circle, parallel to either of the two parallel sides, and terminated by the other two sides, is equal to a fourth part of the perimeter of the figure.

33. With a given point as centre describe a circle to cut a given circle at right angles. How must the point be situated that this may be possible?

34. If two circles touch and a chord be drawn through the point of contact, the tangents at the other points where the chord meets the circles are parallel.

35. From a given point  $A$  without a circle whose centre is  $O$  draw a straight line cutting the circle at the points  $B$  and  $C$ , so that the area  $BOC$  may be the greatest possible.

36. Two circles  $PAB$ ,  $QAB$  cut one another at  $A$ : it is required to draw through  $A$  a straight line  $PQ$  so that  $PQ$  may be equal to a given straight line.

37. Two given circles touch one another externally in the point  $P$ , and are touched by the line  $AB$  in the points  $A$  and  $B$  respectively. Shew that the circle described on  $AB$  as diameter passes through the point  $P$ , and touches the line joining the centres of the two given circles.

38. From a point without a circle draw a line such that the part of it included within the circle may be of a given length less than the diameter of the circle.

39. When an equilateral polygon is described about a circle it must necessarily be equiangular if the number of sides be odd, but not otherwise.

40. One circle touches another internally at the point  $A$ : it is required to draw from  $A$  a straight line such that the part of it between the circles may be equal to a given straight line, which is not greater than the difference between the diameters of the circles.

41. If a hexagon circumscribe a circle, the sums of its alternate sides are equal.

42. If a polygon of an even number of sides circumscribe a circle, the sum of the alternate sides is equal to half the perimeter of the polygon.

43. Under what condition is it possible to describe a circle to touch the two diagonals and two opposite sides of a quadrilateral?

44. If a parallelogram be circumscribed about a circle with centre  $O$ , and a line be drawn touching the circle and cutting the sides of the parallelogram, or the sides produced in  $A, B, C, D$ , prove that the angles  $AOB, BOC$ , and  $COD$  are, each of them, equal to a half of one or other of the angles of the parallelogram.

45. If two equal circles be placed at such a distance apart that the tangent drawn to either of them from the centre of the other is equal to a diameter, they will have a common tangent equal to the radius.

46. Draw a straight line to touch one given circle so that the part of it contained by another given circle shall be equal to a given straight line not greater than the diameter of the latter circle.

47.  $AB$  is the diameter and  $C$  the centre of a semicircle: shew that  $O$  the centre of any circle inscribed in the semicircle is equidistant from  $C$  and from the tangent to the semicircle parallel to  $AB$ .

48. A circle is drawn to touch a given circle and a given straight line. Shew that the points of contact are always in the same straight line with one or other of two fixed points in the circumference of the given circle.

49. Describe a circle to touch a given circle and a given straight line.

How many solutions are there?

50. A series of circles is described passing through one angular point of a parallelogram and a fixed point on the diagonal through that angular point. Shew that the sum of the squares on the tangents from the extremities of the other diagonal is the same for each circle of the series.

51. A quadrilateral is bounded by a diameter of a circle, the tangents at its extremities, and a third tangent: shew that its area is equal to half that of the rectangle contained by the diameter and the side opposite to it.



52. If the centres of two circles which touch each other externally be fixed, the external common tangents of those circles will touch another circle of which the straight line joining the fixed centres is the diameter.

53.  $TP, TQ$  are tangents from  $T$  to two circles which meet in  $A$  and  $PQ$  meets the circles in  $P', Q'$ , and the angles  $PAP', QAQ'$  are equal; find the locus of  $T$ .

54. Describe a circle of given radius passing through a given point and touching a given straight line.

How many solutions may there be?

55. Describe a circle of given radius touching a given straight line and a given circle.

How many solutions may there be?

56. Given one angle of a triangle, the side opposite it, and the sum of the other two sides, construct the triangle.

57.  $AOB, COD$  are two diameters of a circle whose centre is  $O$ , and they are mutually perpendicular. If  $P$  be any point on the circumference, shew that  $CP$  and  $DP$  are the internal and external bisectors of the angle  $APB$ .

58. If  $AB$  and  $CD$  be two perpendicular diameters of a circle and  $P$  any point on the arc  $ACB$ , shew that  $D$  is equally distant from  $PA, PB$ .

59. If two circles intersect each other, prove that each common tangent subtends, at the two common points, angles which are supplementary to each other.

60. From one of the points of intersection of two equal circles, each of which passes through the centre of the other, a straight line is drawn to intersect the circles in two other points: prove that these points form an equilateral triangle with the other point of intersection of the two circles.

61. A series of circles touch a fixed straight line at a fixed point: shew that the tangents at the points where they cut a parallel fixed straight line all touch a fixed circle.

62. Two points  $P, Q$  are taken in two arcs described on the same straight line  $AB$ , and on the same side of it; the angles  $PAQ, PBQ$  are bisected by the straight lines  $AR, BR$  meeting at  $R$ : prove that the angle  $ARB$  is constant.

63.  $APB$  is a fixed chord passing through  $P$  a point of intersection of two circles  $AQP, PBR$ ; and  $QPR$  is any other chord of the circles passing through  $P$ : shew that  $AQ$  and  $RB$  when produced meet at a constant angle.

64.  $A, B, C, D$  are four points on a circle. Prove that the four points where the perpendiculars from any point  $O$  to the straight lines  $AB$  and  $CD$  meet  $AC$  and  $BD$  lie on a circle.

65. Any number of triangles are described on the same base  $BC$ , and on the same side of it having their vertical angles equal, and perpendiculars, intersecting at  $D$ , are drawn from  $B$  and  $C$  on the opposite sides; find the locus of  $D$ .

66. Let  $O$  and  $C$  be any fixed points on the circumference of a circle, and  $OA$  any chord; then, if  $AC$  be joined and produced to  $B$ , so that  $OB$  is equal to  $OA$ , the locus of  $B$  is an equal circle.

67. If  $ABC$  be a triangle,  $AD$  and  $BE$  the perpendiculars from  $A$  and  $B$  upon  $BC$  and  $AC$ ,  $DF$  and  $EG$  the perpendiculars from  $D$  and  $E$  upon  $AC$  and  $BC$ , then  $FG$  is parallel to  $AB$ .

68. The four circles which pass through the middle points of the sides of the four triangles formed by two sides of a quadrilateral and one of its two diagonals intersect in a point.

69. In a circle two chords of given length are drawn so as not to intersect, and one of them is fixed in position; the opposite extremities of the chords are joined by straight lines intersecting within the circle: shew that the locus of the point of intersection will be a portion of the circumference of a circle, passing through the extremities of the fixed chord.

70. The centre  $C$  of a circle  $BPQ$  lies on another circle  $APQ$  of which  $PBA$  is a diameter. Prove that  $PC$  is parallel to  $BQ$ .

71. Through one of the points of intersection of two circles, centres  $A$  and  $B$ , a chord is drawn meeting the circles at  $P$  and  $Q$  respectively. The lines  $PA$ ,  $QB$  intersect in  $C$ . Find the locus of  $C$ .

72. At each extremity of the base of a triangle a straight line is drawn making with the base an angle equal to half the sum of the two base angles; prove that these lines will meet on the external bisector of the vertical angle.

73. In the figure of Euclid i. 47 the feet of the perpendiculars drawn from the centre of the largest square upon the three sides of the given right angled triangle are collinear.

74. Three points  $A$ ,  $B$ ,  $C$  are taken on a circle. The tangents at  $B$  and  $C$  meet in  $T$ . If from  $T$  a straight line be drawn parallel to  $AB$ , it meets  $AC$  in the diameter perpendicular to  $AB$ .

75. If  $ABC$  be a triangle inscribed in a circle,  $PQ$  a diameter, and perpendiculars be let fall from  $P$  on the two sides meeting in  $A$ , and from  $Q$  on those meeting in  $B$ , the lines joining the feet of the two sets of perpendiculars will intersect at right angles.

76. Through any point  $O$  on the side  $BC$  of an equilateral triangle  $ABC$ ,  $OK$ ,  $OL$  are drawn parallel to  $AB$ ,  $AC$  respectively to meet  $AC$ ,  $AB$  respectively in  $K$  and  $L$ : the circle through  $O$ ,  $K$  and  $L$  cuts  $AB$ ,  $AC$  again in  $P$  and  $Q$ . Prove that  $OPQ$  is an equilateral triangle.

77. Two chords  $AB, CD$  of a circle intersect in  $E$ . From  $EB, ED$ , produced if necessary, parts  $EF, EG$  are cut off respectively equal to  $ED, EB$ . Prove that  $FG$  is parallel to  $CA$ .

78. If  $ABC$  be an equilateral triangle and  $D$  be any point on the circumference of the circle  $ABC$ , then one of the three distances  $DA, DB, DC$  is equal to the sum of the other two.

79. If through  $E$  a point of intersection of two circles  $ACE, BDE$  two straight lines  $AB$  and  $CD$  be drawn terminated by the circles, then  $AC$  and  $BD$  cut one another at a constant angle.

80. If  $AD, BE, CF$  be the perpendiculars from the angles  $A, B, C$  of a triangle on the opposite sides, these lines bisect the angles of the triangle  $DEF$ .

81.  $AB$  is a common chord of the segments  $ACB, ADEB$  of two circles, and through  $C$  any point on  $ACB$  are drawn the straight lines  $ACE, BCD$ : prove that the arc  $DE$  is of invariable length.

82. Divide a circle into two parts so that the angle contained in one arc shall be equal to twice the angle contained in the other.

83. A quadrilateral is inscribed in a circle: shew that the sum of the angles in the four arcs of the circle exterior to the quadrilateral is equal to six right angles.

84.  $A, B, C, D$  are four points taken in order on the circumference of a circle; the straight lines  $AB, CD$  produced intersect at  $P$ , and  $AD, BC$  at  $Q$ : shew that the straight lines which bisect the angles  $APC, AQC$  are perpendicular to each other.

85. A quadrilateral can have one circle inscribed in it and another circumscribed about it: shew that the straight lines joining the opposite points of contact of the inscribed circle are perpendicular to each other.

86. If  $D, E, F$  be three points on the sides  $BC, CA, AB$  of a triangle, the perimeter of the triangle  $DEF$  is least when  $D, E, F$  are the feet of the perpendiculars from  $A, B, C$  on  $BC, CA, AB$ .

87. Shew that the four straight lines bisecting the angles of any quadrilateral form a quadrilateral which can be inscribed in a circle.

88. If a polygon of six sides be inscribed in a circle, the sum of three alternate angles is equal to four right angles.

89.  $AB$  is a diameter of a circle,  $CD$  a chord perpendicular to it. A straight line through  $A$  cuts the circle in  $E$ , and  $CD$  produced in  $F$ : prove that the angles  $AEC, DEF$  are equal.

90. If  $O$  be a point within a triangle  $ABC$ , and  $OD, OE, OF$  be drawn perpendicular to  $BC, CA, AB$ , respectively, the angle  $BOC$  is equal to the sum of the angles  $BAC, EDF$ .

91.  $AOB, COD$  are two chords of a circle which intersect within the circle at  $O$ . Through the point  $A$  a straight line  $AF$  is drawn to meet the tangent to the circle at the point  $C$ , so that the angle  $AFC$  is equal to the angle  $BOC$ . Prove that  $OF$  is parallel to  $BC$ .

92. The sums of the alternate angles of any polygon of an even number of sides inscribed in a circle are equal.

93. Through a point  $C$  in the circumference of a circle two straight lines  $ACB, DCE$  are drawn cutting the circle at  $B$  and  $E$ : shew that the straight line which bisects the angles  $ACE, DCB$  meets the circle at a point equidistant from  $B$  and  $E$ .

94.  $AOB, COD$  are two diameters of the circle  $ABCD$ , at right angles to each other. Equal lengths  $OE, OF$ , are measured off along  $OA, OD$  respectively. Shew that  $BF$  produced cuts  $DE$  at right angles, and that these two straight lines, when produced, intercept between them one fourth of the circumference of the circle.

95. If the extremities of the chord of a circle slide upon two straight lines, which intersect on the circumference, every point in the circumference will trace out a straight line.

96. Any point  $P$  is taken on a given arc of a circle described on a line  $AB$ , and perpendiculars  $AG$  and  $BH$  are dropped on  $BP$  and  $AP$  respectively; prove that  $GH$  touches a fixed circle.

97.  $OA$  and  $OB$  are two fixed radii of a given circle at right angles to each other and  $POQ$  is a variable diameter; prove that the locus of the intersection of  $PA$  and  $QB$  is a circle equal to the given one.

98. Describe a circle touching a given straight line at a given point, such that the tangents drawn to it from two given points in the straight line may be parallel.

99. Describe a circle with a given radius touching a given straight line, such that the tangents drawn to it from two given points in the straight line may be parallel.

100. Two circles intersect at the points  $A$  and  $B$ , from which are drawn chords to a point  $C$  in one of the circumferences, and these chords, produced if necessary, cut the other circumference at  $D$  and  $E$ : shew that the straight line  $DE$  cuts at right angles that diameter of the circle  $ABC$  which passes through  $C$ .

101. If squares be described externally on the sides and the hypotenuse of a right-angled triangle, the straight line joining the intersection of the diagonals of the latter square with the right angle is perpendicular to the straight line joining the intersections of the diagonals of the two former.

102.  $C$  is the centre of a given circle,  $CA$  a straight line less than the radius; find the point of the circumference at which  $CA$  subtends the greatest angle.

103. If two chords of a circle meet at a right angle within or without a circle, the squares on their segments are together equal to the square on the diameter.

104. On a given base  $BC$  a triangle  $ABC$  is described such that  $AC$  is equal to the perpendicular from  $B$  upon  $AC$ . Find the locus of the vertex  $A$ .

105. Draw from a given point in the circumference of a circle, a chord which shall be bisected by its point of intersection with a given chord of the circle.

106. Through any fixed point of a chord of a circle other chords are drawn; shew that the straight lines from the middle point of the first chord to the middle points of the others will meet them all at the same angle.

107. The two angles at the base of a triangle are bisected by two straight lines on which perpendiculars are drawn from the vertex: shew that the straight line which passes through the feet of these perpendiculars will be parallel to the base and will bisect the sides.

108. A point  $P$  is taken on a circle of which  $AB$  is a fixed chord; a parallelogram is described of which  $AB$  and  $AP$  are adjacent sides: find the locus of the middle points of the diagonals of the parallelogram.

Find the maximum length of the diagonal drawn through  $A$ .

109.  $A$  is a fixed point on a circle whose centre is  $O$  and  $BOD$  is a diameter. The tangents at  $A$  and  $D$  meet in  $L$ . Shew that the locus of the intersection of  $LB$  with the perpendicular from  $A$  on  $OB$  is a circle.

110.  $ABC$  is a triangle inscribed in a circle;  $AD$ ,  $BE$ , perpendiculars to  $BC$  and  $AC$ , meet in  $O$ :  $AK$  is a diameter of the circle: prove that  $CK$  is equal to  $BO$ .

111. A straight line touches a circle at the point  $P$  and  $QR$  is a chord of a second circle parallel to this tangent;  $PQ$ ,  $PR$  cut the first circle in  $S$ ,  $T$ , and the second circle in  $U$ ,  $V$ ; prove that  $ST$ ,  $UV$  are parallel to each other.

112.  $A$ ,  $B$ ,  $C$  are three points on a circle, the bisectors of the angle  $BAC$  and the angle between  $BA$  produced and  $AC$  meet  $BC$  and  $BC$  produced in  $E$  and  $F$  respectively; shew that the tangent at  $A$  bisects  $EF$ .

113. A circle is described passing through the right angle of a right-angled triangle, and touching the hypotenuse at its middle point: prove that the arc of this circle, cut off by either side of the triangle, is divided at the middle point of the hypotenuse into two parts one of which is double of the other.

114. If through the point of contact  $P$  of two circles two straight lines  $PQq$ ,  $PRr$  be drawn to meet the circles in  $Q$ ,  $R$ , and  $q$ ,  $r$  respectively; then the triangles  $PQR$ ,  $Pqr$  are equiangular.

115. The angle between two chords of a circle, which intersect within the circle, is equal to half the sum of the angles subtended at the centre by the intercepted arcs.

116.  $ABCD$  is a parallelogram and any straight line is drawn cutting the sides  $AB$ ,  $CD$  in  $P$ ,  $Q$  respectively. The circle passing through  $A$ ,  $P$ , and  $Q$  cuts  $AD$  in  $S$ , and  $AC$  in  $T$ . Shew that the circles circumscribing the triangle  $DSQ$ ,  $CTQ$  touch one another at the point  $Q$ .

117. If one circle touch another internally, any chord of the greater circle which touches the less is divided at the point of its contact into segments which subtend equal angles at the point of contact of the two circles.

118. Two circles are drawn touching a circle, whose centre is  $C$ , in  $P$  and  $Q$  respectively and intersecting in  $PQ$  produced, and again in  $R$ . Prove that the angles  $CRP$ ,  $CRQ$  are equal.

119. The angle between two chords of a circle, which intersect when produced without the circle, is equal to half the difference of the angles subtended at the centre by the intercepted arcs.

120. Construct a triangle, having given the base, the vertical angle, and the length of the straight line drawn from the vertex to the middle point of the base.

121.  $A$  is a given point: it is required to draw from  $A$  two straight lines which shall contain a given angle and intercept on a given straight line a part of given length.

122.  $A$  and  $B$  are two points,  $XY$  a given straight line of unlimited length, not passing through  $A$  or  $B$ ; find the point (or points) in  $XY$  at which the straight line  $AB$  subtends an angle equal to a given angle.

Also state when this problem is impossible.

123. The chords of two intersecting circles which are bisected at any point of the common chord are equal.

124. Find a point in the tangent to a circle at one end of a given diameter, such that when a straight line is drawn from this point to the other extremity of the diameter, the rectangle contained by the part of it without the circle and the part within the circle may be equal to a given square not greater than that on the diameter.

125. If perpendiculars be drawn from the extremities of the diameter of a circle upon any chord or any chord produced, the rectangle under the perpendiculars is equal to that under the segments between the feet of the perpendiculars and either extremity of the chord.

126. Two circles intersect and any straight line  $ACBD$  cuts them in  $A$ ,  $B$  and  $C$ ,  $D$  respectively. If  $E$  be a point on the line such that the rectangles contained by  $AC$ ,  $BE$  and  $BD$ ,  $CE$  are equal, the locus of  $E$  is a straight line.

127.  $ABD$  is an isosceles triangle having the side  $AB$  equal to the side  $BD$ ;  $AC$  is drawn at right angles to  $AB$  to meet  $BD$  produced in  $C$ , and the bisector of the angle  $B$  meets  $AC$  in  $O$ . Shew that the square on  $AB$  is equal to the difference of the rectangles  $CB, BD$  and  $CA, AO$ .

128. Through a given point without a circle draw, when possible, a straight line cutting the circle so that the part within the circle may be equal to the part without the circle.

What condition is necessary in order that a real solution may be possible?

129. Two equal circles have their centres at  $A$  and  $B$ :  $O$  is a fixed point outside those circles.  $A$  is the centre of a third circle whose radius is equal to  $OB$ : prove that a right-angled triangle can be constructed having its sides equal to the tangents from  $O$  to the three circles.

130. Prove that, if a straight line drawn parallel to the side  $BC$  of a triangle  $ABC$  cut the sides  $AB, AC$  in  $D, E$  respectively, the rectangle contained by  $AB, AE$  is equal to the rectangle contained by  $AC, AD$ .

131. Construct a triangle, having given the base, the vertical angle, and the length of the straight line drawn from the vertex to the base bisecting the vertical angle.

132. Through two given points on the circumference of a given circle draw the two parallel chords of the circle which shall contain the greatest rectangle.

133. A straight line  $PAQ$  is drawn through  $A$  one of the points of intersection of two given circles to meet the circles again in  $P$  and  $Q$ : find the line which makes the rectangle  $AP, AQ$  a maximum.

134. Produce a given straight line so that the rectangle contained by the whole line thus produced, and the part produced, shall be equal to the square on another given line.

135. Two circles intersect in  $O$ , and through  $O$  a straight line  $ORS$  is drawn cutting the circles again in  $R$  and  $S$ .  $SO$  is produced to  $P$ , so that the rectangle  $OP, RS$  is constant. Shew that the locus of  $P$  is a straight line.

136. If from the foot of the perpendicular from each of the angular points of a triangle on the opposite side a perpendicular be drawn to each of the other sides, the feet of the six perpendiculars so drawn lie on a circle.

137.  $AB, CD$  are chords of a circle intersecting at  $O$ , and  $AC, DB$  meet at  $P$ . If circles be described about the triangles  $AOC, BOD$ , the angles between their tangents at  $O$  will be equal to the angles at  $P$ , and their other common point will lie on  $OP$ .

138. If  $ABC$  be a triangle,  $D, E, F$  the feet of the perpendiculars from  $A, B, C$  on the opposite sides,  $O$  their point of intersection, the rectangles  $DO, DA$ , and  $DE, DF$  are equal.

139. A straight line  $AD$  is drawn bisecting the angle  $A$  of a triangle  $ABC$  and meeting the side  $BC$  in  $D$ . Find a point  $E$  in  $BC$  produced either way such that the square on  $ED$  may be equal to the rectangle contained by  $EB, EC$ .

140. If  $D, D'; E, E'; F, F'$  be pairs of points on the sides  $BC, CA, AB$  of a triangle respectively, such that  $D, D', E, E'$  are concyclic,  $E, E', F, F'$  are concyclic,  $F, F', D, D'$  are concyclic, then  $D, D', E, E', F, F'$  are concyclic.

141. In the straight line  $PQ$  a point  $R$  is taken, and circles are described on  $PR, RQ$  as diameters; shew how to draw a line through  $P$  such that the chords intercepted by the two circles may be equal.

142. If  $M$  be the middle point of  $PQ$ , where  $P$  and  $Q$  are points without a given circle, the sum of the squares on the tangents to the circle from  $P$  and  $Q$  is equal to twice the sum of the square on the tangent from  $M$  and the square on  $PM$ .

143. A circle  $FDG$  touches another circle  $BDE$  in  $D$  and a chord  $AB$  of the latter in  $F$ : prove that the rectangle  $FA, FB$  is equal to the rectangle contained by  $CE$  and the diameter of  $FDG$ , where  $CE$  is drawn perpendicular to  $AB$  at its middle point  $C$  and on the same side of it as the circle  $FDG$ .

144. Given the base and the vertical angle of a triangle, prove that the locus of the centre of the nine-point circle is a circle.

145. If a circle be circumscribed to a triangle, the middle point of the base is equally distant from the orthocentre and the point diametrically opposite the vertex. Also these three points are in the same straight line.



## BOOK IV.

### DEFINITIONS.

#### DEFINITION 1.

*A figure of five sides is called a pentagon,  
one of six sides is called a hexagon,  
one of eight sides is called an octagon,  
one of ten sides is called a decagon,  
one of twelve sides is called a dodecagon\*.*

**DEFINITION 2.** *When each of the angular points of one rectilineal figure lies on one of the sides of a second rectilineal figure, and each of the sides of the second figure passes through one of the angular points of the first figure, the first figure is said to be inscribed in the second figure, and the second figure is said to be described about the first figure.*

\* Derived from πέντε "five," ἑξ "six," ὀκτώ "eight," δέκα "ten," δώδεκα "twelve," respectively, and γωνία "an angle."

## PROPOSITION 1.

*To draw a chord of a given circle equal to a given straight line.*

Let  $ABC$  be the given circle, and  $D$  the given straight line:

it is required to draw a chord of the circle  $ABC$  equal to  $D$ .

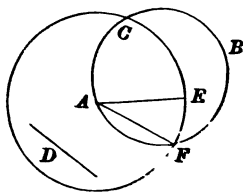
CONSTRUCTION. Take any point  $A$  on the circle  $ABC$ , and from  $A$  draw  $AE$  equal to  $D$ . (I. Prop. 2.)

If  $E$  lie on the circle  $ABC$ , what is required is done, for in the circle  $ABC$  the chord  $AE$  is drawn equal to  $D$ .

But if  $E$  do not lie on the circle  $ABC$ , with  $A$  as centre and  $AE$  as radius describe the circle  $ECF$  cutting the circle  $ABC$  at  $F$ .

Draw  $AF$ .

$AF$  is a chord drawn as required.



PROOF. Because  $A$  is the centre of the circle  $ECF$ ,

$AF$  is equal to  $AE$ .

But  $AE$  is equal to  $D$ .

(Constr.)

Therefore  $AF$  is equal to  $D$ ,

and it is a chord of the circle  $ABC$ .

Wherefore, a chord  $AF$  of the given circle  $ABC$  has been drawn equal to the given straight line  $D$ .

It is clear that it is not possible to draw a chord of a given circle to be equal to a given straight line, if the given line be greater than the diameter of the circle (III. Prop. 8, Part 1); and further that, if a solution be possible, in general *two* chords can be drawn from a given point equal to the given line.

In the diagram, if the two circles intersect in  $C$ , the chord  $AC$  also is equal to the given line.

## EXERCISES.

1. In a given circle draw a chord parallel to one given straight line and equal to another.

2. On a given circle find a point such that, if chords be drawn to it from the extremities of a given chord, their sum shall be equal to a given straight line.

How many solutions are there in the different cases which may occur?

## PROPOSITION 2.

*To inscribe in a given circle a triangle equiangular to a given triangle.*

Let  $ABC$  be the given circle, and  $DEF$  the given triangle :

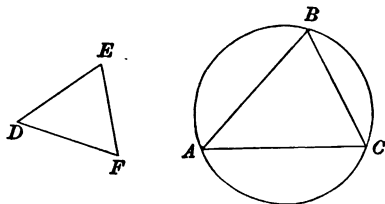
it is required to inscribe in the circle  $ABC$  a triangle equiangular to the triangle  $DEF$ .

CONSTRUCTION. Take any point  $A$  on the circle, and through  $A$  draw the chord  $AB$  to cut off the arc  $ACB$  containing an angle equal to the angle  $DFE$ , and through  $A$  draw the chord  $AC$  to cut off the arc  $ABC$  containing an angle equal to the angle  $DEF$ .\*

(III. Prop. 34.)

Draw  $BC$  :

the triangle  $ABC$  is inscribed as required.



PROOF. Because the arc  $ACB$  contains an angle equal to the angle  $DFE$ , (Constr.)

and the angle  $ACB$  is contained by the arc  $ACB$ , the angle  $ACB$  is equal to the angle  $DFE$ .

Similarly it can be proved that

the angle  $ABC$  is equal to the angle  $DEF$ .

And because the sum of three angles of a triangle is equal to two right angles, (I. Prop. 32.)

and the angles  $ACB, ABC$  are equal to the angles  $DFE, DEF$  respectively,

\* It must be noticed that the arcs  $ACB, ABC$  are measured in opposite directions along the circumference from the point  $A$ .

the remaining angle  $BAC$  of the triangle  $ABC$  is equal to the remaining angle  $EDF$  of the triangle  $DEF$ ; therefore the triangle  $ABC$  is equiangular to the triangle  $DEF$ .

Wherefore, *a triangle  $ABC$  equiangular to the triangle  $DEF$  has been inscribed in the given circle  $ABC$ .*

Since the arc  $ABC$  may be measured in either direction along the circumference from  $A$ , we see that two triangles equiangular to a given triangle can be inscribed in a given circle so as to have a vertical angle equal to a given angle of the triangle at a given point on the circle, and that six triangles equiangular to a given triangle can be inscribed in a given circle, so as to have one of its vertical angles at the given point on the circle.

#### EXERCISES.

1. Prove that all triangles inscribed in the same circle equiangular to each other are equal in all respects.

2. The altitude of an equilateral triangle is equal to a side of an equilateral triangle inscribed in a circle described on one of the sides of the original triangle as diameter.

3.  $ABC, A'B'C'$  are two triangles equiangular to each other inscribed in a circle  $AA'BB'CC'$ . The pairs of sides  $BC, B'C'$ ;  $CA, C'A'$ ;  $AB, A'B'$  intersect in  $a, b, c$  respectively.

Prove that the triangle  $abc$  is equiangular to the triangle  $ABC$ .

## PROPOSITION 3.

*To describe about a given circle a triangle equiangular to a given triangle.*

Let  $ABC$  be the given circle and  $DEF$  the given triangle:

it is required to describe about the circle  $ABC$  a triangle equiangular to the triangle  $DEF$ .

CONSTRUCTION. Find the centre  $G$  of the circle  $ABC$ ,  
(III. Prop. 5.)

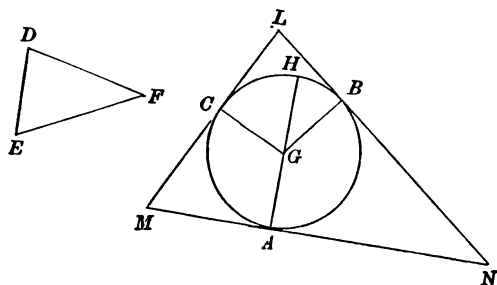
and draw any diameter  $HGA$  meeting the circle in  $A$ .

At  $G$  in  $GH$  draw the straight lines  $GB, GC$  on opposite sides of  $GH$  making the angles  $BGH, CGH$  equal to the angles  $EFD, DEF$ ,  
(I. Prop. 23.)

meeting the circle in  $B, C$ .

Through  $A, B, C$  draw  $MAN, NBL, LCM$  at right angles to  $GA, GB, GC$  respectively:  
(I. Prop. 11.)

the triangle  $LMN$  is a triangle described as required.



PROOF. Because the sum of the angles of the quadrilateral  $GBNA$  is equal to four right angles,

(I. Prop. 32, Coroll.)

and two of the angles  $GANA, GBN$  are right angles,  
the sum of the angles  $AGB, ANB$  is equal to two right angles.

But the sum of the angles  $AGB, HGB$  is equal to two right angles;  
(I. Prop. 13.)

therefore the sum of the angles  $AGB, ANB$  is equal to the sum of the angles  $AGB, HGB$ ;

therefore the angle  $ANB$  is equal to the angle  $HGB$ ,  
that is, to the angle  $EFD$ .

Similarly it can be proved that

the angle  $LMN$  is equal to the angle  $DEF$ ;  
therefore the remaining angle  $NLM$  of the triangle  $LMN$   
is equal to the remaining angle  $EDF$  of the triangle  $DEF$ .  
(I. Prop. 32.)

Therefore the triangle  $LMN$  is equiangular to the triangle  $DEF$ .

Again, because  $MN$  is drawn through  $A$  a point on the circle  $ABC$  at right angles to the radius  $AG$ ,

$MN$  touches the circle. (III. Prop. 16.)

Similarly it can be proved that  $NL$ ,  $LM$  touch the circle.  
Therefore the triangle  $LMN$  is described about the circle  $ABC$ .

Wherefore, *a triangle  $LMN$  equiangular to the given triangle  $DEF$  has been described about the given circle  $ABC$ .*

#### EXERCISES.

1. Prove that all triangles described about the same circle equiangular to each other are equal in all respects.

2. Describe a triangle about a given circle to have its sides parallel to the sides of a given triangle.

How many solutions are there?

3. The angles of the triangle formed by joining the points of contact of the inscribed circle of a triangle with the sides are equal to the halves of the supplements of the corresponding angles of the original triangle.

4. If  $ABC$ ,  $A'B'C'$  be two equal triangles described about a circle in the same sense and the pairs of sides  $BC$ ,  $B'C'$ ;  $CA$ ,  $C'A'$ ;  $AB$ ,  $A'B'$  meet in  $a$ ,  $b$ ,  $c$  respectively,  $a$ ,  $b$ ,  $c$  are equidistant from the centre of the circle.

## PROPOSITION 4.

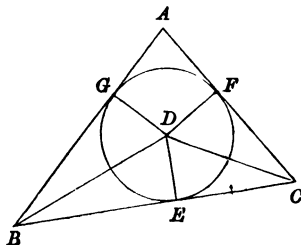
*To inscribe a circle in a given triangle.*

Let  $ABC$  be the given triangle:  
it is required to inscribe a circle in the triangle  $ABC$ .

CONSTRUCTION. Bisect two of the angles  $ABC$ ,  $BCA$  of the triangle  $ABC$  by  $BD$ ,  $CD$  meeting at  $D$ , (I. Prop. 9.) and from  $D$  draw  $DE$ ,  $DF$ ,  $DG$  perpendicular to  $BC$ ,  $CA$ ,  $AB$  respectively. (I. Prop. 12.)

With  $D$  as centre and  $DE$ ,  $DF$ , or  $DG$  as radius describe a circle:

it will be a circle described as required.



PROOF. Because in the triangles  $DEB$ ,  $DGB$ ,  
the angle  $DBE$  is equal to the angle  $DBG$ , (Constr.)  
and the angle  $DEB$  is equal to the angle  $DGB$ ,  
(I. Prop. 10 B.)

and the side  $BD$  is common,  
the triangles are equal in all respects;  
(I. Prop. 26, Part 2.)

therefore  $DE$  is equal to  $DG$ .

Similarly it can be proved that  $DE$  is equal to  $DF$ .  
Therefore the three straight lines  $DE$ ,  $DF$ ,  $DG$  are equal to one another, and the circle described with  $D$  as centre, and  $DE$ ,  $DF$ , or  $DG$  as radius passes through the extremities of the other two;  
and touches the straight lines  $BC$ ,  $CA$ ,  $AB$ , because the angles at the points  $E$ ,  $F$ ,  $G$  are right angles, and the



straight line drawn through a point on a circle at right angles to the radius touches the circle. (III. Prop. 16.) Therefore the straight lines  $AB$ ,  $BC$ ,  $CA$  do each of them touch the circle, and therefore the circle is inscribed in the triangle  $ABC$ .

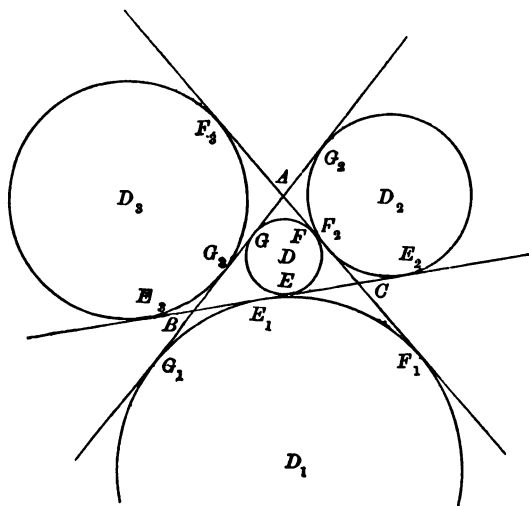
Wherefore, *the circle  $EFG$  has been inscribed in the given triangle  $ABC$ .*

## EXERCISES.

1. The base of a triangle is fixed, and the vertex describes a circle passing through the extremities of the base: find the locus of the centre of the inscribed circle.
2. If a polygon be described about a circle, the bisectors of all its angles meet in a common point.
3. Describe a circle to touch a given circle and two given tangents to the circle.
4. Construct a triangle, having given the base, the vertical angle and the radius of the inscribed circle.
5. Find the centre of a circle cutting off three equal chords from the sides of a triangle.
6. The triangle whose vertices are the three points of contact of the inscribed circle with the sides of a triangle, is always acute-angled.

It can be proved in the same manner as in Proposition 4 that, if the angles at  $B$  and  $C$  of the triangle be bisected *externally* by  $BD_1$ ,  $CD_1$ , meeting at  $D_1$ , and perpendiculars,  $D_1E_1$ ,  $D_1F_1$ ,  $D_1G_1$ , be drawn, the circle described with  $D_1$  as centre and either of the three lines  $D_1E_1$ ,  $D_1F_1$ ,  $D_1G_1$  as radius will touch the three sides of the triangle. Such a circle satisfies the definition (III. Def. 10) of an inscribed circle.

The circles are however generally distinguished thus, the circle  $EFG$ , which lies wholly within the triangle  $ABC$ , is called **the inscribed circle**, whereas the circle  $E_1F_1G_1$  is called an **escribed circle**, and is said to be **escribed beyond the side  $BC$** , to distinguish it from the two other circles which can, in a similar manner, be escribed beyond  $CA$  and beyond  $AB$  respectively.



## EXERCISES.

1. Prove that the radius of the inscribed circle of a triangle is less than the radius of any one of the escribed circles.
2. Prove that the greatest of the escribed circles of a triangle is that which is escribed beyond the greatest side, and the least, beyond the least side.
3. If the centres of the escribed circles of a triangle be joined, and the points of contact of the inscribed circle be joined, the two triangles so formed are equiangular to each other.
4. A circle touches the side  $BC$  of a triangle  $ABC$  and the other two sides produced: shew that the distance between the points of contact of the side  $BC$  with this circle and with the inscribed circle, is equal to the difference between the sides  $AB$  and  $AC$ .
5. Construct a triangle, having given its base, one of the angles at the base, and the distance between the centre of the inscribed circle and the centre of the circle touching the base and the sides produced.
6. Prove that, if  $A, B$  be two fixed points on a circle and  $P$  a variable point, the locus of the centre of each of the escribed circles of the triangle  $APB$  is a circle.
7. The centre of the inscribed circle of a triangle is the ortho-centre of the triangle formed by the centres of the escribed circles.

## PROPOSITION 5.

*To describe a circle about a given triangle.*

Let  $ABC$  be the given triangle :  
it is required to describe a circle about the triangle  $ABC$ .

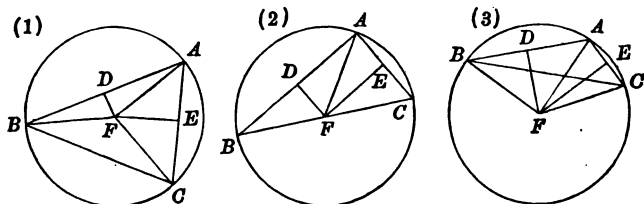
CONSTRUCTION. Bisect two of the sides  $AB$ ,  $AC$  of the triangle  $ABC$ , at  $D$ ,  $E$ , (I. Prop. 10.)  
and draw  $DF$ ,  $EF$  at right angles to  $AB$ ,  $AC$  meeting at  $F$ . (I. Prop. 12.)

Draw  $FA$ , and with  $F$  as centre and  $FA$  as radius describe a circle:

this is a circle described as required.

$F$  must lie either in  $BC$  (fig. 2) or not in  $BC$  (figs. 1 and 3).

If  $F$  do not lie in  $BC$ , draw  $FB$ ,  $FC$ .



PROOF. Because in the triangles  $FDA$ ,  $FDB$ ,  
 $AD$  is equal to  $BD$ , (Constr.)

and  $DF$  is equal to  $DF$ ,

and the angle  $ADF$  is equal to the angle  $BDF$ , (Constr.)  
the triangles are equal in all respects; (I. Prop. 4.)  
therefore  $FA$  is equal to  $FB$ .

Similarly it can be proved that  $FA$  is equal to  $FC$ .

Therefore the circle described with  $F$  as centre and  $FA$  as radius passes through the points  $B$  and  $C$ , and is described about the triangle  $ABC$ .

Wherefore, a circle  $ABC$  has been described about the given triangle  $ABC$ .

The construction of Proposition 5 shews that only one circle can be described about a given triangle, a theorem which has already been established otherwise.

(III. Prop. 9, Coroll. 2.)

The circle  $ABC$  is often spoken of as the **circumscribed circle** of the triangle  $ABC$ .

Propositions 4 and 5 solve problems of the same nature; each shews how to describe a circle to satisfy three given conditions. The problem of Proposition 4 *to describe a circle to touch three given straight lines*, admits of 4 solutions; Proposition 5, *to describe a circle to pass through three given points*, admits of but a single solution.

A circle can be described to satisfy three (and not more than three) independent conditions, but it will be found that the solution is not always unique: if the problem be one which can be solved by geometrical methods, the number of solutions will be found to be 1 or 2 or  $4=2 \times 2$  or  $8=2 \times 2 \times 2$  or some higher power of 2.

The number 2 occurs in one of its powers from the fact that at each step of the solution where choice is possible, the choice lies between the *two* intersections of a circle and a straight line or the *two* intersections of two circles.

If it be required to describe a circle to touch four or more given straight lines, or to pass through four or more given points, relations of some kind must exist between the positions of the lines or of the points in order that a solution may be possible.

### EXERCISES.

1. Inscribe in an equilateral triangle another equilateral triangle having each side equal to a given straight line.
2. Shew how to cut off the corners of an equilateral triangle, so as to leave a regular hexagon.
3. The sides  $AB$ ,  $AC$  of a triangle are produced and the exterior angles are bisected by straight lines meeting in  $O$ : if a circle be described about the triangle  $BOC$ , its centre will be on the circle described about the triangle  $ABC$ .

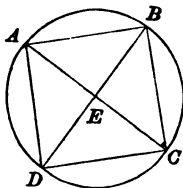
## PROPOSITION 6.

*To inscribe a square in a given circle.*

Let  $ABCD$  be the given circle:  
it is required to inscribe a square in the circle  $ABCD$ .

CONSTRUCTION. Find the centre  $E$  of the circle  $ABCD$ ,  
(III. Prop. 5.)  
and draw two diameters  $AEC$ ,  $BED$  at right angles to one  
another. (I. Prop. 11.)

Draw  $AB$ ,  $BC$ ,  $CD$ ,  $DA$ :  
the quadrilateral  $ABCD$  is a square inscribed as required.



PROOF. Because the angle  $BEC$  is double of the angle  $BAC$ ,

and the angle  $AED$  is double of the angle  $ACD$ ,  
(III. Prop. 20.)

and the angle  $BEC$  is equal to the angle  $AED$ ,  
(I. Prop. 15.)

therefore the angle  $BAC$  is equal to the angle  $ACD$ .

And because  $AC$  meeting  $AB$ ,  $CD$  makes the alternate  
angles  $BAC$ ,  $ACD$  equal,

$AB$ ,  $CD$  are parallel. (I. Prop. 27.)

Similarly it can be proved that  $AD$ ,  $BC$  are parallel.

Therefore the quadrilateral  $ABCD$  is a parallelogram.

Again, because  $ABC$  is an angle in a semicircle  $ABC$ ,  
the angle  $ABC$  is a right angle. (III. Prop. 31.)

Therefore the parallelogram  $ABCD$  is a rectangle.  
(I. Def. 19.)

Again, because in the triangles  $AEB$ ,  $CEB$ ,

$AE$  is equal to  $CE$ ,

$BE$  to  $BE$ ,

and the angle  $AEB$  to the angle  $CEB$ ,

the triangles are equal in all respects ; (I. Prop. 4.)

therefore  $BA$  is equal to  $BC$ .

Therefore the rectangle  $ABCD$  is a square.

(I. Def. 20.)

Wherefore, a square  $ABCD$  has been inscribed in the given circle  $ABCD$ .

#### EXERCISES.

1. Inscribe a regular octagon in a given circle.
2. Shew how to cut off the corners of a square so as to leave a regular octagon.
3. Inscribe in a given square, a square to have its sides equal to a given straight line.

## PROPOSITION 7.

*To describe a square about a given circle.*

Let  $ABCD$  be the given circle :

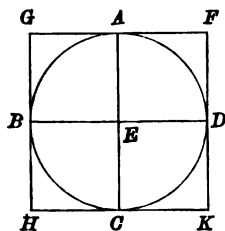
it is required to describe a square about it.

CONSTRUCTION. Find  $E$  the centre of the circle  $ABCD$ ,  
(III. Prop. 5.)

and draw two diameters  $AEC$ ,  $BED$  at right angles to one  
another. (I. Prop. 11.)

Draw  $GAF$ ,  $HCK$  parallel to  $BD$ , and  $GBH$ ,  $FDK$  parallel  
to  $AC$ :

the quadrilateral  $FGHK$  is a square described as required.



PROOF. Because  $GF$ ,  $HK$  are each parallel to  $BD$ ,

$GF$ ,  $HK$  are parallel to each other. (I. Prop. 30.)

Similarly it can be proved that

$GH$ ,  $FK$  are parallel to each other;

therefore  $FGHK$  is a parallelogram.

Again, because  $GAEB$  is a parallelogram, (Constr.)

the angle  $AGB$  is equal to the angle  $AEB$ ,  
(I. Prop. 34.)

which is a right angle; (Constr.)

therefore the parallelogram  $FGHK$  is a rectangle.

(I. Def. 19.)

Again, because  $GBDF$  is a parallelogram,

$GF$  is equal to  $BD$ , a diameter of the circle.

Similarly it can be proved that

$GH$  is equal to  $AC$ , a diameter of the circle;

therefore  $GF$  is equal to  $GH$ .

Therefore the rectangle  $FGHK$  is a square. (I. Def. 20.)



Again, because  $AE$  intersects the parallel lines  $GF$ ,  $BD$ ,  
the angle  $GAE$  is equal to the alternate angle  $AED$ ,

(I. Prop. 29.)

which is a right angle ; (Constr.)

therefore  $GA$  touches the circle. (III. Prop. 16.)

Similarly it can be proved that  $GB$ ,  $HC$ ,  $KD$  touch the circle.

Wherefore, a square  $FGHK$  has been described about the given circle  $ABCD$ .

## EXERCISES.

1. Describe a regular octagon about a given circle.
2. Prove that the area of a circumscribed square of a circle is double that of an inscribed square.
3. If two circles be such that the same square can be inscribed in one and described about the other, the circles must be concentric.  
Is any other condition necessary?
4. If a parallelogram admit of a circle being inscribed in it and another circle being described about it, the parallelogram must be a square.

## PROPOSITION 8.

*To inscribe a circle in a given square.*

Let  $ABCD$  be the given square :

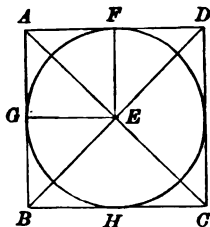
it is required to inscribe a circle in it.

CONSTRUCTION. Draw  $AC$ ,  $BD$  intersecting in  $E$ .

From  $E$  draw  $EF$ ,  $EG$  perpendicular to  $DA$ ,  $AB$  two  
of the sides of the square, (I. Prop. 12.)

and with  $E$  as centre and  $EF$  or  $EG$  as radius describe  
a circle :

it is a circle inscribed as required.



PROOF. Because  $CD$  is equal to  $AD$ ,

(I. Prop. 34, Coroll. 1.)

the angle  $CAD$  is equal to the angle  $ACD$ . (I. Prop. 5.)

And because  $AC$  meets the parallels  $AB$ ,  $DC$ ,  
the angle  $BAC$  is equal to the alternate angle  $ACD$  ;

(I. Prop. 29.)

therefore the angle  $BAC$  is equal to the angle  $DAC$ .

And because in the triangles  $GAE$ ,  $FAE$ ,

the angle  $GAE$  is equal to the angle  $FAE$ ,

and the angle  $AGE$  to the angle  $AFE$ ,

and  $AE$  equal to  $AE$ ,

the triangles are equal in all respects ;

(I. Prop. 26, Part 2.)

therefore  $EG$  is equal to  $EF$ ,

and therefore the circle described with  $E$  as centre and  
 $EF$  or  $EG$  as radius passes through the extremity of the  
other, and touches the two sides  $DA$ ,  $AB$ . (III. Prop. 16.)

Similarly it can be proved that this circle touches each of the sides  $BC, CD$ :

it is therefore inscribed in the square  $ABCD$ .

Wherefore, *a circle  $FGH$  has been inscribed in the given square  $ABCD$ .*

#### EXERCISES.

1. Prove that a circle can be inscribed in any rhombus.
2. Two opposite sides of a convex quadrilateral are together equal to the other two. Shew that a circle can be inscribed in the quadrilateral.
3.  $AD, BE$  are common tangents to two circles  $ABC, DEC$ , that touch each other; shew that a circle may be inscribed in the quadrilateral  $ABED$ , and a circle may be described about it.

## PROPOSITION 9.

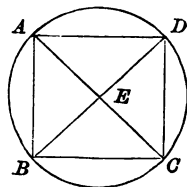
*To describe a circle about a given square.*

Let  $ABCD$  be the given square:

it is required to describe a circle about it.

CONSTRUCTION. Draw  $AC$ ,  $BD$  intersecting at  $E$ ;  
and with  $E$  as centre and  $EA$ ,  $EB$ ,  $EC$  or  $ED$  as radius  
describe a circle :

it is a circle described as required.



PROOF. Because in the triangles  $BAC$ ,  $DAC$ ,  
 $BA$  is equal to  $DA$ , (I. Prop. 34, Coroll. 1.)  
and  $BC$  to  $DC$ ,  
and  $AC$  to  $AC$ ,

the triangles are equal in all respects; (I. Prop. 8.)  
therefore the angle  $BAC$  is equal to the angle  $DAC$ ,  
or the angle  $BAC$  is half of the angle  $BAD$ .

Similarly it can be proved that

the angle  $ABD$  is half of the angle  $ABC$ .

But the angle  $BAD$  is equal to the angle  $ABC$ ;

(I. Prop. 29, Coroll. and I. Prop. 10 B.)

therefore the angle  $BAE$  is equal to the angle  $ABE$ .

Therefore  $BE$  is equal to  $AE$ . (I. Prop. 6.)

Similarly it can be proved that  $CE$  and  $DE$  are each of  
them equal to  $AE$  or  $BE$ .

Therefore the circle described with  $E$  as centre and one of  
four lines  $EA$ ,  $EB$ ,  $EC$ , or  $ED$  as radius passes through  
the extremities of the other three, and is described about  
the square  $ABCD$ .

Wherefore, a circle  $ABCD$  has been described about the  
given square  $ABCD$ .

## EXERCISES.

1. A point is taken without a square such that the angles subtended at it by three sides of the square are equal: shew that the locus of the point is the circumference of the circle circumscribing the square.

2. Find the locus of a point at which two given sides of a square subtend equal angles.

3. If a quadrilateral be capable of having a quadrilateral of minimum perimeter inscribed in it, it must admit of a circle being inscribed in it.

4.  $ABCD$  is a quadrilateral inscribed in a circle, and its diagonals,  $AC$ ,  $BD$  intersect at right angles in  $E$ ;  $K$ ,  $L$ ,  $M$ ,  $N$  are the feet of the perpendiculars from  $E$  on the sides of the quadrilateral. Shew that  $KLMN$  can have circles inscribed in it and described about it.

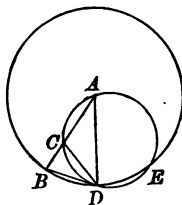
## PROPOSITION 10.

*To construct a triangle having each of two angles double of the third angle.*

CONSTRUCTION. Take any straight line  $AB$ , and divide it at  $C$ , so that the rectangle  $AB, BC$  may be equal to the square on  $AC$ . (II. Prop. 11.)

With centre  $A$  and radius  $AB$  describe the circle  $BDE$ , and draw a chord  $BD$  equal to  $AC$ , (Prop. 1.) and draw  $DA$ :

the triangle  $ABD$  is a triangle constructed as required. Draw  $DC$ , and about the triangle  $ACD$  describe the circle  $ACD$ . (Prop. 5.)



PROOF. Because the rectangle  $AB, BC$  is equal to the square on  $AC$ , and  $AC$  is equal to  $BD$ ,

the rectangle  $AB, BC$  is equal to the square on  $BD$ .

And because from the point  $B$  without the circle  $ACD$ ,  $BCA$  is drawn cutting it in  $C$  and  $A$ , and  $BD$  is drawn meeting it in  $D$ ,

and the rectangle  $AB, BC$  is equal to the square on  $BD$ , therefore  $BD$  touches the circle  $ACD$ . (III. Prop. 37.)

And because  $BD$  touches the circle  $ACD$ , and  $DC$  is drawn from the point of contact  $D$ ,

the angle  $BDC$  is equal to the angle  $DAC$ . (III. Prop. 32.)

To each of these add the angle  $CDA$ ; then the whole angle  $BDA$  is equal to the sum of the angles  $CDA, DAC$ .

But the angle  $BCD$  is equal to the sum of the angles  $CDA, DAC$ ; (I. Prop. 32.)

therefore the angle  $BDA$  is equal to the angle  $BCD$ .

But because  $AD$  is equal to  $AB$ ,  
 the angle  $ABD$  is equal to the angle  $ADB$ ; (I. Prop. 5.)  
 therefore the angle  $DBA$  is equal to the angle  $BCD$ .  
 And because the angle  $DBC$  is equal to the angle  $BCD$ ,  
 $DC$  is equal to  $DB$ . (I. Prop. 6.)  
 But  $DB$  is equal to  $CA$ ; (Constr.)  
 therefore  $CA$  is equal to  $CD$ ;  
 therefore the angle  $CDA$  is equal to the angle  $CAD$ . (I. Prop. 5.)  
 Therefore the sum of the angles  $CAD, CDA$  is double of the  
 angle  $CAD$ .  
 But the angle  $BCD$  is equal to the sum of the angles  
 $CAD, CDA$ ; (I. Prop. 32.)  
 therefore the angle  $BCD$  is double of the angle  $CAD$ .  
 And the angle  $BCD$  has been proved equal to each of the  
 angles  $BDA, DBA$ ;  
 therefore each of the angles  $BDA, DBA$  is double of the  
 angle  $BAD$ .

Wherefore, a triangle  $ABD$  has been constructed having  
 each of two angles  $ABD, ADB$  double of the third angle  
 $BAD$ .

It will be observed that the smaller angle of the triangle constructed  
 in this proposition is equal to a fifth of two right angles.

## EXERCISES.

1. Describe an isosceles triangle having each of the angles at the base one third of the vertical angle.
2. Divide a right angle into five equal parts.
3. In the figure of Proposition 10, if the two circles cut again at  $E$ , then  $DE$  is equal to  $DC$ .
4. In the figure of Proposition 10, the circle  $ACD$  is equal to the circle described about the triangle  $ABD$ .
5. In the figure of Proposition 10, if  $AF$  be the diameter of the smaller circle,  $DF$  is equal to a radius of the circle which circumscribes the triangle  $BCD$ .
6. If in the figure of Proposition 10, the circles meet again in  $E$ , then  $CE$  is parallel to  $BD$ .

## PROPOSITION 11.

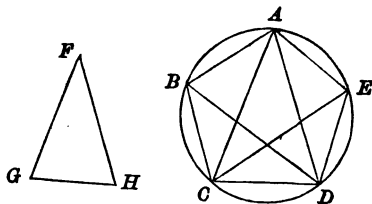
*To inscribe a regular pentagon in a given circle.*

Let  $ABCDE$  be the given circle :  
it is required to inscribe a regular pentagon in the circle  $ABCDE$ .

CONSTRUCTION. Construct a triangle  $FGH$  having each of the angles at  $G, H$  double of the angle at  $F$ ; (Prop. 10.) in the circle  $ABCDE$ , inscribe the triangle  $ACD$ , equiangular to the triangle  $FGH$ , so that the angles  $CAD, ADC, DCA$  may be equal to the angles  $GFH, FHG, HGF$  respectively. (Prop. 2.)

Bisect the angles  $ACD, ADC$  by the straight lines  $CE, DB$ ,  
(I. Prop. 9.)

and draw  $CB, BA, AE, ED$  :  
then  $ABCDE$  is a pentagon inscribed as required.



PROOF. Because each of the angles  $ACD, ADC$  is double of the angle  $CAD$ , (Constr.)

and they are bisected by  $CE, DB$ ,  
the five angles  $ADB, BDC, CAD, DCE, ECA$  are equal.  
But equal angles at the circumference of a circle stand on equal arcs; (III. Prop. 26, Coroll.)

therefore the five arcs  $AB, BC, CD, DE, EA$  are equal.  
And the chords, by which equal arcs are subtended, are equal; (III. Prop. 29, Coroll.)

therefore the five straight lines  $AB, BC, CD, DE, EA$  are equal;

therefore the pentagon  $ABCDE$  is equilateral.

Again, the arc  $ED$  is equal to the arc  $BA$ ;  
to each of these add the arc  $AE$ ;



then the whole arc  $AED$  is equal to the whole arc  $BAE$ .  
And the angle  $AED$  is contained by the arc  $AED$ , and  
the angle  $BAE$  by the arc  $BAE$ ;

therefore the angle  $AED$  is equal to the angle  $BAE$ .

(III. Prop. 27, Coroll.)

Similarly it can be proved that each of the angles  $ABC$ ,  
 $BCD$ ,  $CDE$  is equal to the angle  $AED$  or  $BAE$ ;

therefore the pentagon  $ABCDE$  is equiangular.

Therefore  $ABCDE$  is a regular pentagon.

Wherefore, a regular pentagon  $ABCDE$  has been inscribed  
in the given circle  $ABCDE$ .

The following is a complete Geometrical construction for inscribing  
a regular decagon or pentagon in a given circle.

Find  $O$  the centre.

Draw two diameters  $AOC$ ,  $BOD$  at  
right angles to one another.

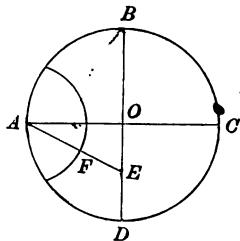
Bisect  $OD$  in  $E$ .

Draw  $AE$  and cut off  $EF$  equal to  
 $OE$ .

Place round the circle ten chords  
equal to  $AF$ .

These chords will be the sides of  
a regular decagon. Draw the chords  
joining five alternate vertices of the decagon; they will be the sides of  
a regular pentagon.

We leave the proof of this as an exercise for the student.



### EXERCISES.

1. A regular pentagon is inscribed in a circle, and alternate angular points are joined by straight lines. Prove that these lines will form by their intersections a regular pentagon.

2. If  $ABCDE$  be a regular pentagon,  $ACEBD$  is a regular star shaped pentagon, each of whose angles is equal to two-fifths of a right angle.

3.  $ABCDE$  is a regular pentagon; draw  $AC$  and  $BD$ , and let  $BD$  meet  $AC$  at  $F$ ; shew that  $AC$  is equal to the sum of  $AB$  and  $BF$ .

4. If  $AB$ ,  $BC$ , and  $CD$  be sides of a regular pentagon, the circle which touches  $AB$  and  $CD$  at  $B$  and  $C$  passes through the centre of the circle inscribed in the pentagon.

## PROPOSITION 12.

*To describe a regular pentagon about a given circle.*

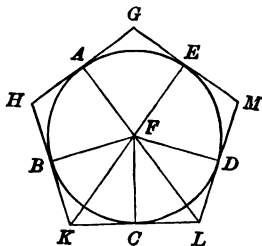
Let  $ABCDE$  be the given circle :  
it is required to describe a regular pentagon about the circle  $ABCDE$ .

CONSTRUCTION. Let  $A, B, C, D, E$  be the angular points of a regular pentagon inscribed in the circle ; (Prop. 11.)  
so that  $AB, BC, CD, DE, EA$  are equal arcs.

Find the centre  $F$ ; (III. Prop. 5.)

draw  $FA, FB, FC, FD, FE$ ,  
and draw  $GAH, HBK, KCL, LDM, MEG$  at right angles  
to  $FA, FB, FC, FD, FE$  respectively; (I. Prop. 11.)  
then  $GHKLM$  is a pentagon described as required.

Draw  $FK, FL$ .



PROOF. Because in the triangles  $FBK, FCK$ ,  
 $FB$  is equal to  $FC$ , and  $FK$  to  $FK$ ,  
and the angle  $FBK$  equal to the angle  $FCK$ ,  
each being a right angle, (I. Prop. 10 B.)  
and each of the angles  $FKB, FKC$  therefore is less than  
a right angle, (I. Prop. 17.)  
the triangles are equal in all respects ;  
(I. Prop. 26 A, Coroll.)  
therefore  $KB$  is equal to  $KC$ ,  
and the angle  $BFK$  equal to the angle  $CFK$ ,  
and the angle  $BKF$  to the angle  $CKF$ ;  
therefore  $KF$  bisects each of the angles  $BFC, BKC$ .  
Similarly it can be proved that  $LF$  bisects each of the  
angles  $CFD, CLD$ .

Again, because the arc  $BC$  is equal to the arc  $CD$ ,  
the angle  $BFC$  is equal to the angle  $CFD$ .

(III. Prop. 27, Coroll.)

And because the angle  $KFC$  is half of the angle  $BFC$ ,  
and the angle  $LFC$  is half of the angle  $CFD$ ;  
therefore the angle  $KFC$  is equal to the angle  $LFC$ .

Now because in the triangles  $KFC$ ,  $LFC$ ,  
the angle  $KFC$  is equal to the angle  $LFC$ ,  
and the angle  $KCF$  to the angle  $LCF$ ,  
and  $FC$  to  $FC$ ,

the triangles are equal in all respects; (I. Prop. 26, Part 1.)  
therefore  $KC$  is equal to  $LC$ ,

and the angle  $FKC$  equal to the angle  $FLC$ .

Now it has been proved that  $KB$  is equal to  $KC$ ,  
and that  $KL$  is double of  $KC$ ;

and it can similarly be proved that  $KH$  is double of  $KB$ ;  
therefore  $HK$  is equal to  $KL$ .

Similarly it can be proved that any two consecutive sides  
of  $GHKLM$  are equal;

therefore the pentagon  $GHKLM$  is equilateral.

And because it has been proved that the angles  $FKC$ ,  $FLC$   
are equal,

and that the angle  $BKC$  is double of the angle  $FKC$ ,

and the angle  $CLD$  double of the angle  $FLC$ ,

therefore the angle  $BKC$  is equal to the angle  $CLD$ .

Similarly it can be proved that any two consecutive angles  
of  $GHKLM$  are equal.

Therefore the pentagon  $GHKLM$  is equiangular.

The pentagon is therefore regular.

And because each side is drawn at right angles to a radius  
of the circle at its extremity, it touches the circle;

(III. Prop. 16.)

therefore the pentagon is described about the circle  $ABCDE$ .

Wherefore, a regular pentagon  $GHKLM$  has been described about the given circle  $ABCDE$ .

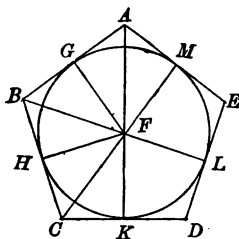
## PROPOSITION 13.

*To inscribe a circle in a given regular pentagon.*

Let  $ABCDE$  be the given regular pentagon :  
it is required to inscribe a circle in  $ABCDE$ .

CONSTRUCTION. Bisect any two consecutive angles of  
the pentagon  $ABC$ ,  $BCD$  by  $BF$ ,  $CF$  (I. Prop. 9.)  
meeting at  $F$ ;  
draw  $FG$ ,  $FH$ ,  $FK$ ,  $FL$ ,  $FM$  perpendicular to  $AB$ ,  $BC$ ,  $CD$ ,  
 $DE$ ,  $EA$  respectively. (I. Prop. 12.)  
With  $F$  as centre and  $FG$ ,  $FH$ ,  $FK$ ,  $FL$  or  $FM$  as radius  
describe a circle:  
it is a circle inscribed as required.

Draw  $AF$ .



PROOF. Because in the triangles  $ABF$ ,  $CBF$ ,  
 $AB$  is equal to  $CB$ , (Hypothesis.)  
and  $BF$  to  $BF$ ,  
and the angle  $ABF$  to the angle  $CBF$ , (Constr.)  
the triangles are equal in all respects;  
(I. Prop. 4.)

therefore  $FA$  is equal to  $FC$ ,  
and the angle  $BAF$  is equal to the angle  $BCF$ .

Again, because the angle  $BAE$  is equal to the angle  $BCD$ ,  
(Hypothesis.)  
and the angle  $BAF$  has been proved equal to the angle  
 $BCF$ ,  
and the angle  $BCF$  is half of the angle  $BCD$ , (Constr.)  
therefore the angle  $BAF$  is half of the angle  $BAE$ ,  
or  $AF$  bisects the angle  $BAE$ .

Similarly it can be proved that  $EF$ ,  $DF$  bisect the angles  $AED$ ,  $EDC$ ;  
therefore the bisectors of all the angles of the pentagon meet in a point.

Again, because in the triangles  $FCH$ ,  $FCK$ ,  
the angle  $FHC$  is equal to the angle  $FKC$ ,  
and the angle  $FCH$  to the angle  $FCK$ , (Constr.)  
and  $FC$  to  $FC$ ,

the triangles are equal in all respects;

(I. Prop. 26, Part 2.)

therefore  $FH$  is equal to  $FK$ .

Similarly it can be proved that the perpendiculars on any two consecutive sides are equal to one another :  
therefore  $FG$ ,  $FH$ ,  $FK$ ,  $FL$ ,  $FM$  are equal, and the circle described with  $F$  as centre and one of the five lines  $FG$ ,  $FH$ ,  $FK$ ,  $FL$  or  $FM$  as radius passes through the extremities of the other four ; and because the angles at  $G$ ,  $H$ ,  $K$ ,  $L$ ,  $M$  are right angles, it touches  $AB$ ,  $BC$ ,  $CD$ ,  $DE$ ,  $EA$ .  
(III. Prop. 16.)

Wherefore, a circle  $GHKLM$  has been inscribed in the given regular pentagon  $ABCDE$ .

### EXERCISES.

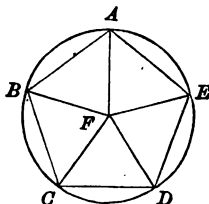
1. How many conditions are necessary in order that a given pentagon may admit of a circle being inscribed in it?
2. Prove that the bisectors of all the angles of any regular polygon meet in a point.

## PROPOSITION 14.

*To describe a circle about a given regular pentagon.*

Let  $ABCDE$  be the given regular pentagon:  
it is required to describe a circle about  $ABCDE$ .

CONSTRUCTION. Bisect any two consecutive angles of the pentagon  $ABC$ ,  $BCD$ , by  $BF$ ,  $CF$  (I. Prop. 9.) meeting at  $F$ , and draw  $FA$ ,  $FE$ ,  $FD$ ; with  $F$  as centre and  $FA$ ,  $FB$ ,  $FC$ ,  $FD$  or  $FE$  a radius describe a circle:  
it will be a circle described as required.



PROOF. Because the angle  $ABC$  is equal to the angle  $BCD$ , (Hypothesis.)  
and the angle  $FBC$  is half of the angle  $ABC$ ,  
and the angle  $FCB$  is half of the angle  $BCD$ ,  
therefore the angle  $FBC$  is equal to the angle  $FCB$ ;  
therefore  $FC$  is equal to  $FB$ . (I. Prop. 6.)

Again, because in the triangles  $ABF$ ,  $CBF$ ,  
 $AB$  is equal to  $CB$ , (Hypothesis.)  
 $BF$  to  $BF$ ,  
and the angle  $ABF$  to the angle  $CBF$ , (Constr.)  
the triangles are equal in all respects;  
(I. Prop. 4.)

therefore  $FA$  is equal to  $FC$ ,  
and the angle  $FAB$  to the angle  $FCB$ .  
And because the angle  $FAB$  is equal to the angle  $FCB$ ,  
and the angle  $BAE$  to the angle  $BCD$ ,  
(Hypothesis.)

and the angle  $FCB$  is half of the angle  $BCD$ ,  
(Constr.)

therefore the angle  $FAB$  is half of the angle  $BAE$ ;  
and  $FA$  bisects the angle  $BAE$ .

Similarly it can be proved that  $FD$ ,  $FE$  bisect the angles  $CDE$ ,  $DEA$  respectively, and that  $FD$  and  $FE$  are each of them equal to  $FA$  or  $FC$ ; therefore  $FA$ ,  $FB$ ,  $FC$ ,  $FD$ ,  $FE$  are all equal, and the circle described with  $F$  as centre and one of the five lines  $FA$ ,  $FB$ ,  $FC$ ,  $FD$ ,  $FE$  as radius passes through the extremities of the other four, and is described about the pentagon  $ABCDE$ .

Wherefore, a circle  $ABCDE$  has been described about the given regular pentagon  $ABCDE$ .

### EXERCISES.

1. Describe a regular decagon to have five of its vertices coincident with those of a given regular pentagon.
2. How many conditions are necessary in order that a given pentagon may admit of a circle being described about it? State the conditions.
3. Shew how to cut off the corners of a regular pentagon so as to leave a regular decagon.

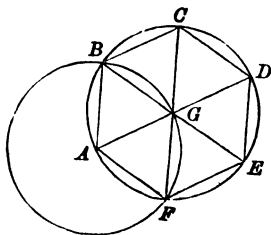
## PROPOSITION 15.

*To inscribe a regular hexagon in a given circle.*

Let  $ABCDEF$  be the given circle :  
it is required to inscribe a regular hexagon in the circle  
 $ABCDEF$ .

CONSTRUCTION. Find  $G$  the centre of the circle ;  
(III. Prop. 5.)

draw any diameter  $AGD$ ,  
and with  $A$  as centre and  $AG$  as radius describe the circle  
 $GBF$  intersecting the circle  $ABCDEF$  in  $B$  and  $F$ .  
Draw  $BG$ ,  $FG$  and produce them to meet the circle again  
in  $E$  and  $C$ , and draw  $AB$ ,  $BC$ ,  $CD$ ,  $DE$ ,  $EF$ ,  $FA$  :  
then  $ABCDEF$  is a hexagon inscribed as required.



PROOF. Because  $G$  is the centre of the circle  $ABCDEF$ ,  
 $GB$  is equal to  $GA$ .

And because  $A$  is the centre of the circle  $BGF$ ,  
 $AB$  is equal to  $AG$ .

Therefore  $AB$ ,  $BG$ ,  $GA$  are all equal.

Therefore the angles  $AGB$ ,  $BAG$ ,  $GBA$  are all equal.

(I. Prop. 5, Coroll. 1.)

But the sum of these three angles is equal to two right  
angles ;

(I. Prop. 32.)

therefore the angle  $AGB$  is equal to one-third of two right  
angles.

Similarly it can be proved that the angle  $FGA$  is equal to  
one-third of two right angles.



But the sum of the three angles  $BGA$ ,  $AGF$ ,  $FGE$  is equal to two right angles; (I. Prop. 13.)

therefore the angle  $FGE$  is one-third of two right angles, and therefore the angles  $BGA$ ,  $AGF$ ,  $FGE$  are equal.

And because opposite vertical angles are equal, (I. Prop. 15.)

the angles opposite to these are equal;

therefore all the angles  $AGF$ ,  $FGE$ ,  $EGD$ ,  $DGC$ ,  $CGB$ ,  $BGA$  are equal.

Therefore the arcs  $AF$ ,  $FE$ ,  $ED$ ,  $DC$ ,  $CB$ ,  $BA$  are equal.

(III. Prop. 26, Coroll.)

Therefore the chords  $AF$ ,  $FE$ ,  $ED$ ,  $DC$ ,  $CB$ ,  $BA$  are equal.

(III. Prop. 29, Coroll.)

Therefore the hexagon  $ABCDEF$  is equilateral.

Again, because the arcs  $BAF$ ,  $AFE$ ,  $FED$ ,  $EDC$ ,  $DCB$ ,  $CBA$  are equal,

the angles  $BAF$ ,  $AFE$ ,  $FED$ ,  $EDC$ ,  $DCB$ ,  $CBA$  in those arcs are equal. (III. Prop. 27, Coroll.)

Therefore the hexagon  $ABCDEF$  is equiangular:

it is therefore regular, and it is inscribed in the circle  $ABCDEF$ .

Wherefore, a regular hexagon  $ABCDEF$  has been inscribed in the given circle  $ABCDEF$ .

#### EXERCISES.

1. Inscribe a regular dodecagon in a given circle.
2. If  $ABCDEF$  be a regular hexagon, and  $AC$ ,  $BD$ ,  $CE$ ,  $DF$ ,  $EA$ ,  $FB$  be drawn, they will form another regular hexagon of one third the area.
3. The perimeter of the inscribed equilateral triangle of a circle is three quarters the perimeter of the circumscribed regular hexagon.
4. Six equal circles can be described each touching a given circle and two of the others.

## PROPOSITION 16.

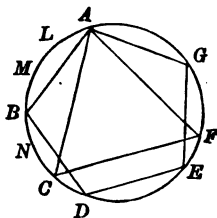
*To inscribe a regular polygon of fifteen sides in a given circle.*

Let  $ABCD$  be the given circle :  
it is required to inscribe a regular polygon of fifteen sides  
in the circle  $ABCD$ .

CONSTRUCTION. Let  $A, C$  be two angular points of an  
equilateral triangle inscribed in the circle, (Prop. 2.)  
and let  $A, B, D$  be three angular points of a regular pentagon  
inscribed in the circle. (Prop. 11.)

Draw  $CD$ ,  
and place round the circle fifteen chords  $AL, LM, \dots$  each  
equal to  $CD$ .

The figure  $ALM \dots$  is a polygon inscribed as required.



PROOF. If the whole circle contain fifteen equal parts,  
the arc  $ABC$ , which is a third of the circle, contains five  
such parts,  
and the arc  $ABD$ , which is made up of the arcs  $AB, BD$ ,  
each of which is a fifth of the circle, contains six such  
parts ;  
therefore the arc  $CD$ , which is the difference of the arcs  
 $ABD, ABC$ , consists of one such part ;  
therefore the arc  $CD$  is one-fifteenth of the circle.  
And because the arcs  $AL, LM, \dots$ , which are the shorter  
arcs cut off by equal chords  $AL, LM, \dots$ , are equal,  
(III. Prop. 28, Coroll.)  
each of the arcs  $AL, LM, \dots$ , is one fifteenth of the circle.

Therefore the extremity of the last chord coincides with the point *A*, and the extremities of the chords which have been placed round the circle exactly divide the circle into fifteen equal parts.

The figure *ALM*... therefore is equilateral.

And as each of the angles is contained by an arc made up of two of the fifteen equal parts,  
all the angles are equal ;

(III. Prop. 27, Coroll.)

therefore the figure *ALM*... is equiangular.

Wherefore, a regular polygon of fifteen sides has been inscribed in the given circle *ABCD*.

#### EXERCISES.

1. Prove that in the figure of Proposition 16 a vertex of the inscribed regular polygon of fifteen sides coincides with each of the vertices of the regular figures used in the construction.

2. Prove that, if all but one of the bisectors of the angles of a polygon meet in a point, they all do so, and a circle can be inscribed in the polygon.

3. Prove that, if all but one of the rectangular bisectors of the sides of a polygon meet in a point, they all do so, and a circle can be described about the polygon.

When a regular polygon of any number of sides is given, we can inscribe a regular polygon of the same number of sides in a given circle, and we can also describe a regular polygon of the same number of sides about a given circle.

Moreover we can always inscribe a circle in a given regular polygon and describe a circle about it.

Methods have now been given for the construction of regular figures of 3, 4, 5, 6, and 15 sides. When any regular polygon is given we can construct a regular polygon of double the number of sides by describing a circle about the polygon and bisecting the smaller arcs subtended by the sides of the given polygon, and so on in succession for each duplication of the number of sides. Thus we see that we can by Euclid's methods construct regular polygons of  $3 \times 2^n$ ,  $4 \times 2^n$ ,  $5 \times 2^n$  and  $15 \times 2^n$  sides, where  $n$  is any positive integer, including zero. It was proved by Gauss\* in the year 1801 that by purely geometrical methods those regular polygons can be constructed, the number of whose sides is a prime† number of the form  $2^n + 1$ . This general law relating to the number of sides includes the case of the triangle ( $n=1$ ) and the pentagon ( $n=2$ ); the next two cases are those of the polygons which have 17 sides ( $n=4$ ) and 257 sides ( $n=7$ ) respectively.

\* *Disquisitiones Arithmeticae* (sectio septima).

† A prime number is an integer which is not divisible without remainder by any integer except itself and unity.

## MISCELLANEOUS EXERCISES.

1. To inscribe a square in a given parallelogram.
2. On a given circle find a point such that, if chords be drawn to it from the extremities of a given chord of the circle, their difference shall be equal to a given straight line less than the given chord.
3.  $AB$  is a fixed chord of a circle whose centre is  $O$ , and  $CD$  is any other chord equal to  $AB$ . The extremities of these chords are joined by straight lines. Prove that, if the joining lines meet each other, their point of intersection lies on the circle which circumscribes the triangle  $AOB$ .
4. Through each angular point of a triangle two straight lines are drawn parallel to the straight lines joining the centre of the circumscribed circle to the other angular points of the triangle. Prove that these six straight lines form an equilateral hexagon, which has three pairs of equal angles.
5. If two equilateral triangles be described about the same circle, they will form an equilateral hexagon whose alternate angles are equal.
6. Describe about a given circle a quadrilateral equiangular to a given quadrilateral.
7. If the diameter of one of the escribed circles of a triangle be equal to the perimeter of the triangle, the triangle is right-angled.
8. The diameter of the inscribed circle of a right-angled triangle is equal to the difference between the sum of the two smaller sides and the hypotenuse.
9. Construct a triangle having given the centres of its inscribed circle and of two of its escribed circles.
10. The side  $AB$  of a triangle  $ABC$  touches the escribed circles at  $G_1, G_2, G_3$ ; prove that  $G_2G_3$  is equal to  $BC$ , and  $G_1G_3$  to  $CA$ .
11. If a circle be inscribed in a right-angled isosceles triangle, the distance from the centre of the circle to the right angle will be equal to the difference between the hypotenuse and a side.

12. Describe a circle to touch each of two given straight lines and to have its centre at a given distance from a third given straight line.

13. Three circles are described, each of which touches one side of a triangle  $ABC$ , and the other two sides produced. If  $D$  be the point of contact of the side  $BC$ ,  $E$  that of  $AC$ , and  $F$  that of  $AB$ , then  $AE$  is equal to  $BD$ ,  $BF$  to  $CE$ , and  $CD$  to  $AF$ .

14. If the diagonals of the quadrilateral  $ABCD$  intersect at right angles at  $O$ , the sum of the radii of the inscribed circles of the triangles  $AOB$ ,  $BOC$ ,  $COD$ ,  $DOA$  is equal to the difference between the sum of the diagonals and the semi-perimeter of the quadrilateral.

15. Having given the hypotenuse of a right-angled triangle and the radius of the inscribed circle, construct the triangle.

16. If the inscribed circle of a triangle  $ABC$  touch the sides  $AB$ ,  $AC$  at the points  $D$ ,  $E$ , and a straight line be drawn from  $A$  to the centre of the circle meeting the circumference at  $G$ , the point  $G$  is the centre of the inscribed circle of the triangle  $ADE$ .

17. Two sides of a triangle whose perimeter is constant are given in position: shew that the third side always touches a fixed circle.

18. The points of contact of the inscribed circle of a triangle are joined; and from the angular points of the triangle so formed perpendiculars are drawn to the opposite sides: shew that the triangle of which the feet of these perpendiculars are the angular points has its sides parallel to the sides of the original triangle.

19. Four triangles are formed by three out of four given points on a given circle: shew that a circle may be described so as to pass through the centres of the inscribed circles of the four triangles.

20. The rectangle of the segments of the hypotenuse of a right-angled triangle made by the point of contact of the inscribed circle is equal to the area of the triangle.

21. If on the sides of any triangle three equilateral triangles be constructed, the centres of the inscribed circles of these triangles are the vertices of an equilateral triangle.

22. Describe three equal circles to touch each other and a given circle.

23. Construct an isosceles triangle, having its base equal to the greater, and the diameter of its inscribed circle equal to the less of two given straight lines.

24. In a given right-angled triangle, the lengths of the sides containing the right angle are 6 and 8 feet respectively. Find the lengths of the segments into which the hypotenuse is divided by the circle inscribed in the triangle.

25. Two triangles  $ABC$ ,  $DEF$  are inscribed in the same circle so that  $AD$ ,  $BE$ ,  $CF$  meet in one point  $O$ ; prove that, if  $O$  be the centre of the inscribed circle of one of the triangles, it will be the orthocentre of the other.

26. A circle  $B$  passes through the centre of another circle  $A$ ; a triangle is described circumscribing  $A$  and having two angular points on  $B$ ; prove that the third angular point is on the line of centres.

27. A circle is escribed to the side  $BC$  of a triangle  $ABC$  touching the other sides in  $F$  and  $G$ . A tangent  $DE$  is drawn parallel to  $BC$  meeting the sides in  $D$ ,  $E$ .  $DE$  is found to be three times  $BC$  in length. Shew that  $DE$  is twice  $AF$ .

28. The sum of the diameters of the inscribed and the circumscribed circles of a right-angled triangle is equal to the sum of the sides containing the right angle.

29. Prove that two circles can be described with the middle point of the hypotenuse of a right-angled triangle as centre to touch the two circles described on the two sides as diameters.

30. The perpendicular from  $A$  on the opposite side  $BC$  of a triangle  $ABC$ , meets the circumference of the circumscribed circle in  $G$ . If  $P$  be the point in which the perpendiculars from the angles upon the opposite sides intersect, then  $PG$  is bisected by  $BC$ .

31. Through  $C$ , the middle point of the arc  $ACB$  of a circle, any chord  $CP$  is drawn, cutting the straight line  $AB$  in  $Q$ . Shew that the locus of the centre of the circle circumscribing the triangle  $BQP$  is a straight line.

32. The distance of the orthocentre of a triangle from any vertex is double of the distance of the centre of the circumscribed circle from the opposite side.

33. Two equilateral triangles  $ABC$ ,  $DEF$  are inscribed in a circle whose centre is  $O$ .  $AC$ ,  $DF$  intersect in  $P$ , and  $AB$ ,  $DE$  in  $Q$ . Prove that either  $POQ$  is a straight line, or a circle can be described about  $APOQD$ .

34. Given the base, the difference of the angles at the base and the radius of the circumscribing circle of a triangle, shew how to construct the triangle.

35. From the vertices of a triangle draw straight lines which shall form an equilateral hexagon whose area is double that of the triangle.

36. If on each side of an acute-angled triangle as base, an isosceles triangle be constructed the sides of each being equal to the radius of the circumscribed circle, the vertices of these triangles form the vertices of a triangle equal in all respects to the original triangle.

37. If  $O$  be the orthocentre of the triangle  $ABC$ , the triangle formed by the centres of the circles  $OBC$ ,  $OCA$ ,  $OAB$  is equal to the triangle  $ABC$  in all respects,

38. The ends of a straight line  $AB$  move along two fixed straight lines in a plane; prove that there is a point, in rigid connection with  $AB$ , which describes a circle.

39. The circle through  $B, C$  and the centre of the circle inscribed in the triangle  $ABC$  meets the sides  $AB, AC$  again in  $E, F$ ; prove that  $EF$  touches the inscribed circle.

40. Let  $ABC$  be a triangle,  $O$  the centre of the inscribed circle, and  $O', O'', O'''$ , the centres of the escribed circles situated in the angles  $A, B, C$ , respectively; prove (1) that the circumscribed circle passes through the middle points of the lines  $OO', OO'', OO'''$ ; (2) that the four points  $O, B, C, O'$  lie on a circle which has its centre on the circumscribed circle; (3) that the points  $O'', B, C, O'''$  lie on a circle whose centre is on the circumscribed circle.

41. The triangle of least perimeter which can be inscribed in a given acute-angled triangle is the triangle formed by joining the feet of the perpendiculars from the angular points on the opposite sides.

42. Describe a circle to touch a given straight line, and pass through two given points.

43. Describe a circle to pass through two given points and cut off from a given straight line a chord of given length.

44. Describe a circle to pass through two given points, so that the tangent drawn to it from another given point may be of a given length.

45. If  $I, O$  be the centres of the inscribed and the circumscribed circles of a triangle  $ABC$ , and if  $AI$  be produced to meet the circumscribed circle in  $F$ , then  $OF$  bisects  $BC$ .

46. If  $I$  be the centre of the inscribed circle of the triangle  $ABC$ , and  $AI$  produced meet the circumscribed circle at  $F, FB, FI$ , and  $FC$  are all equal.

47. Construct a triangle having given one angular point and the centres of the inscribed and the circumscribed circles.

48.  $O$  is the centre of the circumscribed circle of a triangle  $ABC$ ;  $D, E, F$  are the feet of the perpendiculars from  $A, B, C$ , on the opposite sides: shew that  $OA, OB, OC$  are respectively perpendicular to  $EF, FD, DE$ .

49. The four circles each of which passes through the centres of three of the four circles touching the sides of a triangle are equal to one another.

50. Construct a triangle having given an angle and the radii of the inscribed and the circumscribed circles.

51. From the vertex of a triangle draw a straight line to the base so that the square on the straight line may be equal to the rectangle contained by the segments of the base.



52. Four triangles are formed by three out of four given straight lines; shew that the circumscribed circles of these triangles all pass through a common point.

53. The straight line joining the middle points of the arcs of a circle cut off by two sides of an inscribed equilateral triangle is trisected by those sides.

54. The perpendicular from an angle of an equilateral triangle on the opposite side is equal to three quarters of the diameter of the circumscribed circle.

55. If the inscribed and the circumscribed circles of a triangle be concentric, the triangle is equilateral.

56. The angle  $C$  of the triangle  $ABC$  is a right angle.  $P$  is the intersection of the diagonals of a square on  $AC$ , and  $Q$  of those of a square on  $BC$ . Prove that the circumscribed circle of the triangle  $ABC$  passes through the intersection of  $PQ$  with a perpendicular to  $AB$  drawn through the middle point of  $AB$ .

57. Describe three circles to touch in pairs at three given points.

58. A rhombus is described about a given rectangle. Prove that its centre coincides with that of the rectangle and that each of its angular points lies either on a fixed straight line or on a fixed circle.

59. Describe an isosceles triangle such that three times the vertical angle shall be four times either of the other angles.

60. If  $ABCDE$  be a regular pentagon, and  $AC, BD$  intersect at  $O$ , then  $AO$  is equal to  $DO$ , and the rectangle  $AC, CO$  is equal to the square on  $BC$ .

61. Prove that the difference of the squares on a diagonal and on a side of a regular pentagon is equal to the rectangle contained by them.

62. If with one of the angular points of a regular pentagon as centre and one of its diagonals as radius a circle be described, a side of the pentagon will be equal to a side of the regular decagon inscribed in the circle.

63. If  $ABCDE$  be a pentagon described about a circle, and  $F$  be the point of contact of  $AB$ , then twice  $AF$  is equal to the difference of the sum of  $AB, AE, CD$  and the sum of  $BC, DE$ .

64.  $AOA', BOB'$  are two diameters of a circle at right angles to one another.  $BO$  is bisected at  $C$  and  $AC$  cuts the circle on  $BO$  as diameter in  $D$  and  $E$ . Circles having  $A$  as centre and  $AD, AE$  as radii are described cutting the original circle in  $F$  and  $F', G$  and  $G'$ : prove that  $A'GFF'G'$  is a regular pentagon.

65. A regular hexagon  $ABCDEF$  is inscribed in a circle. A second circle is described through  $A$  and  $B$  to cut the first circle at right angles: and a third circle is described through  $A$  and  $C$  to cut the first circle at right angles. Prove that the diameter of the third circle is three times that of the second.

66. If the alternate angles of an equilateral hexagon be equal to one another, a circle can be inscribed in the hexagon.

67. Construct a regular polygon of  $2n$  sides of equal perimeter with a given regular polygon of  $n$  sides.

68. Any equilateral figure which is inscribed in a circle is also equiangular.

69. Prove that a polygon which is described about one and inscribed in another of two concentric circles must be regular.

70. Shew that it is always possible to describe about a circle a polygon equiangular to any given polygon. Will the two polygons be necessarily similar?

71. Find the locus of the centre of the circumscribing circle of a triangle, when the vertical angle and the sum of the sides containing it are given.

72. Given the circumscribed circle, an escribed circle and the centre of the inscribed circle, construct the triangle.

73. Given an angular point, the circumscribing circle and the orthocentre, construct the triangle.

74. Describe a square about a given quadrilateral.  
How many solutions are there?

75. Construct a square so that each side shall touch one of four given circles.  
How many solutions are there?

## BOOK V.

### DEFINITIONS.

**DEFINITION 1.** *If one magnitude be equal to another magnitude of the same kind repeated twice, thrice or any number of times, the first is said to be a **multiple** of the second, and the second is said to be an **aliquot part** or a **measure** of the first.*

If one magnitude  $A$  be equal to  $m$  times another magnitude  $B$  of the same kind ( $m$  being an integer, i.e. a whole number),  $A$  is said to be the  $m^{\text{th}}$  multiple of  $B$ , and  $B$  the  $m^{\text{th}}$  part of  $A$ .

If  $A$  be any multiple of  $B$  and if  $C$  be the same multiple of  $D$ , then  $A$  and  $C$  are said to be **equimultiples** of  $B$  and  $D$ .

The magnitudes treated of in Book V. are not necessarily Geometrical magnitudes: but they are assumed to be such that any magnitude can be supposed to be repeated as often as desired, in other words, that any multiple we please of a magnitude can be taken. They are assumed also to be such that any one taken twice is greater than it is alone; such quantities as those which are called in Algebra either negative or imaginary are excluded from consideration.

The capital letters  $A, B, C, D$  &c. will be used to denote magnitudes, and the small letters  $m, n, p, q$  &c. to denote whole numbers.

When a magnitude  $A$  is spoken of, the letter  $A$  is supposed to represent the magnitude itself.

**DEFINITION 2.** *The relation of one magnitude to another of the same kind with respect to the multiples of the second or of aliquot parts of the second, which the first is greater than, equal to, or less than, is called the **ratio** of the first to the second.*

It is difficult to convey a precise idea of "ratio" by a definition. The student will gradually acquire a firmer grasp of the meaning of the term as he proceeds. It is important to bear in mind that the difference between two magnitudes is not their ratio.

Whenever the ratio of one magnitude to another is spoken of, it is necessarily implied, although it may not always be expressly stated, that the two magnitudes are of the same kind.

*In the ratio of one magnitude to another, the first is called the antecedent and the second the consequent of the ratio.*

In the ratio of  $A$  to  $B$ ,  $A$  is the antecedent and  $B$  the consequent.

*The ratio of a magnitude to an equal magnitude is called a ratio of equality, and is said to be equal to unity;*

*the ratio of a magnitude to a less magnitude is called a ratio of greater inequality, and is said to be greater than unity;*

*the ratio of a magnitude to a greater magnitude is called a ratio of less inequality, and is said to be less than unity.*

The ratio of one diameter to another diameter of the same circle is a ratio of equality:

the ratio of a diagonal to a side of a square is a ratio of greater inequality:

the ratio of the area of a circle to the area of a square described about the circle is a ratio of less inequality.

**DEFINITION 3.** *If one magnitude repeated any number of times be greater than, equal to, or less than a second magnitude repeated any other number of times, the ratio of the first magnitude to the second magnitude is said to be greater than, equal to, or less than the ratio of the second number to the first number.*

If  $A$ ,  $B$  be two magnitudes such that  $m$  times  $A$  is equal to  $n$  times  $B$ , the ratio of  $A$  to  $B$  is equal to the ratio of  $n$  to  $m$ .

Similarly, if  $m$  times  $A$  be greater than  $n$  times  $B$ , the ratio of  $A$  to  $B$  is greater than that of  $n$  to  $m$ ; and if  $m$  times  $A$  be less than  $n$  times  $B$ , the ratio of  $A$  to  $B$  is less than that of  $n$  to  $m$ .

**DEFINITION 4.** *When two magnitudes of the same kind are such that some measure of the first is equal to some measure of the second, the two magnitudes are said to be commensurable.*

*Two magnitudes of the same kind, which are not commensurable, are said to be incommensurable.*

If  $A, B, C$  be three magnitudes of the same kind such that  $C$  is the  $m^{\text{th}}$  part of  $A$  and  $C$  is the  $n^{\text{th}}$  part of  $B$ , or, in other words, such that  $A$  is equal to  $m$  times  $C$  and  $B$  is equal to  $n$  times  $C$ , then  $A$  and  $B$  have a common measure  $C$ , and therefore are commensurable.

If the ratio of  $A$  to  $B$  be equal to the ratio of one integer to another, say that of  $n$  to  $m$ , and the ratio of  $C$  to  $D$  be also equal to that of  $n$  to  $m$ , the ratio of  $A$  to  $B$  is equal to that of  $C$  to  $D$ : and similarly, if the ratio of  $A$  to  $B$  be equal to that of  $n$  to  $m$ , and the ratio of  $C$  to  $D$  be greater or less than that of  $n$  to  $m$ , the ratio of  $A$  to  $B$  is less or greater respectively than that of  $C$  to  $D$ .

A complete method is thus afforded of testing the equality or the inequality of the ratios of pairs of commensurable magnitudes: but the same method is not applicable to incommensurable magnitudes.

Now it will be manifest from what has been said that, if we have four magnitudes  $A, B, C, D$ , of which  $A$  and  $B$  are incommensurable, the ratio of  $C$  to  $D$  cannot be equal to that of  $A$  to  $B$ , unless  $C$  and  $D$  be also incommensurable.

It is possible to find two magnitudes of the same kind that are not commensurable. It can be proved that a diagonal and a side of the same square are such a pair of magnitudes, and also that the circumference and a diameter of the same circle are another such pair of magnitudes. The question arises how the ratios of two pairs of such incommensurable magnitudes are to be compared.

It is easy to prove that a diagonal of a square is greater than once and less than twice a side; these inequalities give a very rough comparison of the lengths of the two lines. It can be proved that 10 times a diagonal is greater than 14 times and less than 15 times a side: these inequalities give a less rough comparison of the lengths. Again, it can be proved that 100 times a diagonal is greater than 141 times and less than 142 times a side: these inequalities give a still less rough comparison of the lengths of the two lines.

These facts are represented by saying that the ratio of a diagonal to a side is greater than the ratio of 1 to 1 and less than that of 2 to 1: greater than the ratio of 14 to 10 and less than that of 15 to 10: greater than the ratio of 141 to 100 and less than that of 142 to 100.

Pairs of ratios of greater and greater numbers might be quoted, between which the ratio of a diagonal to a side always lies: but no two numbers can be found such that the ratio in question is equal to the ratio of the numbers.

In the following definition of the equality of two ratios, the case of incommensurable magnitudes is included as well as that of commensurable magnitudes.

**DEFINITION 5.** *If four magnitudes be such that, when any equimultiples whatever of the first and the third are taken, and also any equimultiples whatever of the second and the fourth, the multiples of the first and the third are simultaneously either both greater than, or both equal to, or both less than the multiples of the second and the fourth respectively, the ratio of the first magnitude to the second is said to be equal to the ratio of the third to the fourth.*

*When the ratio of the first of four magnitudes to the second is equal to that of the third to the fourth, the magnitudes are said to be **proportionals** or **in proportion**.*

*When four magnitudes are proportionals, the first is said to be to the second as the third to the fourth.*

Let  $A, B, C, D$  be four magnitudes, of which  $A$  and  $B$  are of the same kind, and  $C$  and  $D$  are of the same kind, and let any equimultiples whatever of  $A$  and  $C$ , say  $m$  times  $A$  and  $m$  times  $C$ , be taken, and any equimultiples whatever of  $B$  and  $D$ , say  $n$  times  $B$  and  $n$  times  $D$ ; then, if  $m$  times  $A$  be greater than  $n$  times  $B$  and also  $m$  times  $C$  greater than  $n$  times  $D$ , or else  $m$  times  $A$  be equal to  $n$  times  $B$  and also  $m$  times  $C$  equal to  $n$  times  $D$ , or else  $m$  times  $A$  be less than  $n$  times  $B$  and also  $m$  times  $C$  less than  $n$  times  $D$ , for every possible pair of whole numbers  $m$  and  $n$ , the ratio of  $A$  to  $B$  is equal to the ratio of  $C$  to  $D$ , and  $A, B, C, D$  are **proportionals**.

The fact that four magnitudes  $A, B, C, D$  are in proportion is denoted by saying that  $A$  has to  $B$  the same ratio that  $C$  has to  $D$ , or that the ratio of  $A$  to  $B$  is equal to that of  $C$  to  $D$ , or that  $A$  is to  $B$  as  $C$  to  $D$ : it is expressed still more concisely by the notation

$$A : B = C : D.$$

**NOTE.** It will be observed that when four magnitudes  $A, B, C, D$

are defined in order as proportionals, i.e.  $A$  is to  $B$  as  $C$  to  $D$ , they are at the same time defined as proportionals also in the three several orders  $B, A, D, C$ ;  $C, D, A, B$ ; and  $D, C, B, A$ ; that is to say, it follows from the definition that, if any one of the four proportions

$$A : B = C : D, \quad B : A = D : C, \quad C : D = A : B, \quad D : C = B : A$$

exist, the other three exist also.

It follows at once from Definition 5 that if, of  $A, B, C, D$ , four magnitudes,  $A$  be equal to  $C$  and  $B$  be equal to  $D$ , the ratio of  $A$  to  $B$  is equal to the ratio of  $C$  to  $D$ ; and further that if, of  $A, B, C$ , three magnitudes,  $A$  be equal to  $B$ , the ratio of  $A$  to  $C$  is equal to the ratio of  $B$  to  $C$ , and also the ratio of  $C$  to  $A$  is equal to the ratio of  $C$  to  $B$ .

**DEFINITION 6.** *When four magnitudes are proportionals, the first and the third, the antecedents, are said to be homologous to one another, and the second and the fourth, the consequents, are also said to be homologous to one another.*

In the proportion  $A$  is to  $B$  as  $C$  to  $D$ , the antecedents  $A$  and  $C$  are homologous to one another and the consequents  $B$  and  $D$  are homologous to one another.

**DEFINITION 7.** *If it be possible to take equimultiples of the first and the third of four magnitudes and equimultiples of the second and the fourth, such that the multiple of the first is greater than that of the second and the multiple of the third not greater than that of the fourth, the ratio of the first to the second is said to be greater than that of the third to the fourth; and the ratio of the third to the fourth is said to be less than the ratio of the first to the second.*

As an example the ratios of the numbers 2 to 3 and 5 to 8 may be taken.

If the 5<sup>th</sup> multiples of the first and the third be taken and the 3<sup>rd</sup> multiples of the second and the fourth, the multiples in order are 10, 9, 25, 24: here 10 is greater than 9 and 25 greater than 24: but equality between the ratios is not thereby established.

If the 11<sup>th</sup> multiples of the first and the third be taken, and the 7<sup>th</sup> multiples of the second and the fourth, the multiples in order are 22, 21, 55, 56 : here 22 is greater than 21 and 55 not greater than 56 : and the fact is established that the ratio of 2 to 3 is greater than that of 5 to 8.

**DEFINITION 8.** *The ratio of the first of three magnitudes of the same kind to the third is said to be compounded of the ratio of the first to the second and the ratio of the second to the third.*

*The ratio of the first of three magnitudes of the same kind to the third is also said to be the ratio of the ratio of the first magnitude to the second to the ratio of the third magnitude to the second.*

If  $A$ ,  $B$ ,  $C$  be three magnitudes of the same kind, the ratio of  $A$  to  $C$  is compounded of the ratio  $A$  to  $B$  and the ratio  $B$  to  $C$ .

Further, the ratio of  $A$  to  $C$  is said to be the ratio of the ratio  $A$  to  $B$  to the ratio  $C$  to  $B$ .

**DEFINITION 9.** *If the first of a number of magnitudes of the same kind be to the second as the second to the third and as the third to the fourth and so on, the magnitudes are said to be in continued proportion.*

*If a number of magnitudes be in continued proportion, the ratio of the first to the third is said to be duplicate of the ratio of the first to the second, and the ratio of the first to the fourth is said to be triplicate of the ratio of the first to the second.*

*If four magnitudes be in proportion, the first and the fourth are called the extremes and the second and the fourth the means of the proportion.*

*If three magnitudes be in continued proportion, the first and the third are called the extremes and the second the mean of the proportion ; also the second is called a mean proportional between the first and the third, and the third is called a third proportional to the first and the second.*



## PROPOSITION 1. [EUCLID ELEM. PROP. 4.]

*If four magnitudes in order be proportionals and any equimultiples of the antecedents be taken, and any equimultiples of the consequents, the four multiples are proportionals in the same order as the magnitudes\*.*

Let the magnitudes  $A, B, C, D$  be proportionals, and let  $m, n$  be two given numbers:

it is required to prove that  $m$  times  $A, n$  times  $B, m$  times  $C, n$  times  $D$  are proportionals.

CONSTRUCTION. Let  $p, q$  be any two numbers, and let  $m$  times  $A$  be  $E, n$  times  $B$  be  $F, m$  times  $C$  be  $G$  and  $n$  times  $D$  be  $H$ .

PROOF. Because  $E, G$  are equimultiples of  $A, C$ ,  
 $p$  times  $E, p$  times  $G$  are equimultiples of  $A, C$ ,  
 no matter what number  $p$  may be ;

and because  $F, H$  are equimultiples of  $B, D$ ,  
 $q$  times  $F, q$  times  $H$  are equimultiples of  $B, D$ ,  
 no matter what number  $q$  may be ;

and because  $A$  is to  $B$  as  $C$  to  $D$ ,  
 $p$  times  $E$  and  $p$  times  $G$   
 are both greater than, both equal to or both less than  
 $q$  times  $F$  and  $q$  times  $H$  respectively,

for all values of  $p$  and  $q$ ; (Def. 5.)

therefore  $E$  is to  $F$  as  $G$  to  $H$ ; (Def. 5.)

that is,  $m$  times  $A$  is to  $n$  times  $B$  as  $m$  times  $C$  to  $n$  times  $D$ .

Wherefore, if four magnitudes, &c.

\* Algebraically. If  $a : b = c : d$ , then  $ma : nb = mc : nd$ .

## PROPOSITION 2. [EUCLID ELEM. PROP. 8.]

*The greater of two magnitudes has to a third magnitude a greater ratio than the less has; and a third magnitude has to the less of two other magnitudes a greater ratio than it has to the greater\*.*

Let  $A, B, C$  be three magnitudes of the same kind, of which  $A$  is greater than  $B$ :  
it is required to prove that  $A$  has to  $C$  a greater ratio than  $B$  has to  $C$ , and that  $C$  has to  $B$  a greater ratio than  $C$  has to  $A$ .

CONSTRUCTION. Let the excess of  $A$  over  $B$  be  $D$ ;  
take the  $m^{\text{th}}$  equimultiples of  $B, D$ , such that each is greater than  $C$ , and of the multiples of  $C$  let the  $p^{\text{th}}$  multiple be the first which is greater than  $m$  times  $B$ , and let  $n$  be the number next less than  $p$ .

PROOF. Because  $m$  times  $B$  is not less than  $n$  times  $C$ ,  
and  $m$  times  $D$  is greater than  $C$ ; (Constr.)  
therefore the sum of  $m$  times  $B$  and  $m$  times  $D$  is greater than the sum of  $n$  times  $C$  and  $C$ ,  
that is,  $m$  times  $A$  is greater than  $p$  times  $C$ ;  
and  $m$  times  $B$  is less than  $p$  times  $C$ ; (Constr.)  
therefore  $A$  has to  $C$  a greater ratio than  $B$  has to  $C$ .  
(Def. 7.)

Next, because  $p$  times  $C$  is less than  $m$  times  $A$ ,  
and  $p$  times  $C$  is greater than  $m$  times  $B$ ,  
therefore  $C$  has to  $B$  a greater ratio than  $C$  has to  $A$ .  
(Def. 7.)

Wherefore, *the greater, &c.*

\* Algebraically. If  $a > b$ , then  $a : c > b : c$  and  $c : b > c : a$ .

## PROPOSITION 3. [EUCLID ELEM. PROP. 9.]

*If the ratio of the first of three magnitudes to the third be equal to the ratio of the second to the third, the first magnitude is equal to the second\*.*

Let  $A, B, C$  be three magnitudes of the same kind such that  $A$  is to  $C$  as  $B$  is to  $C$ :

it is required to prove that  $A$  is equal to  $B$ .

PROOF. Because, any magnitude greater than  $B$  has to  $C$  a greater ratio than  $B$  has to  $C$ , (Prop. 2.)  
and  $A$  has to  $C$  the same ratio as  $B$  to  $C$ ,  
 $A$  cannot be greater than  $B$ .

Again, because any magnitude less than  $B$  has to  $C$  a less ratio than  $B$  to  $C$ , (Prop. 2.)  
and  $A$  has to  $C$  the same ratio as  $B$  to  $C$ ,  
 $A$  cannot be less than  $B$ .

Therefore  $A$  must be equal to  $B$ .

Wherefore, *if the ratio of the first, &c.*

\* Algebraically. If  $a : c = b : c$ , then  $a = b$ .

## PROPOSITION 4. PART 1. [EUCLID ELEM. PROP. 10.]

*If the ratio of the first of three magnitudes to the third be greater than the ratio of the second to the third, the first magnitude is greater than the second\*.*

Let  $A, B, C$  be three magnitudes of the same kind, such that  $A$  has to  $C$  a greater ratio than  $B$  to  $C$ :

it is required to prove that  $A$  is greater than  $B$ .

PROOF. Because the ratio of  $A$  to  $C$  is greater than that of  $B$  to  $C$ ,

there are some equimultiples, say the  $m^{\text{th}}$  multiples, of  $A, B$  and some multiple, say the  $n^{\text{th}}$  multiple, of  $C$ , such that  $m$  times  $A$  is greater than  $n$  times  $C$ ,  
and  $m$  times  $B$  not greater than  $n$  times  $C$ ; (Def. 7.)  
therefore there is some number  $m$  such that  $m$  times  $A$  is greater than  $m$  times  $B$ ;

therefore  $A$  is greater than  $B$ .

Wherefore, *if the ratio of the first, &c.*

## PROPOSITION 4. PART 2. [EUCLID ELEM. PROP. 10.]

*If the ratio of the first of three magnitudes to the second be greater than the ratio of the first to the third, the second magnitude is less than the third†.*

Let  $A, B, C$  be three magnitudes of the same kind, such that  $A$  has to  $B$  a greater ratio than  $A$  to  $C$ :

it is required to prove that  $B$  is less than  $C$ .

PROOF. Because the ratio of  $A$  to  $B$  is greater than that of  $A$  to  $C$ , there is some multiple, say the  $m^{\text{th}}$  multiple, of  $A$ , and there are some equimultiples, say the  $n^{\text{th}}$  multiples, of  $B, C$ , such that  $m$  times  $A$  is greater than  $n$  times  $B$ ,

and  $m$  times  $A$  not greater than  $n$  times  $C$ ; (Def. 7.)  
therefore there is some number  $n$  such that  $n$  times  $C$  is greater than  $n$  times  $B$ ;

therefore  $C$  is greater than  $B$ .

Wherefore, *if the ratio of the first, &c.*

\* Algebraically. If  $a : c > b : c$ , then  $a > b$ .

† Algebraically. If  $a : b > a : c$ , then  $b < c$ .

## PROPOSITION 5. [EUCLID ELEM. PROP. 11.]

*Ratios, which are equal to the same ratio, are equal to one another\*.*

Let  $A, B, C, D, E, F$  be six magnitudes, such that

$A$  is to  $B$  as  $C$  to  $D$ ,

and also  $E$  is to  $F$  as  $C$  to  $D$ :

it is required to prove that

$A$  is to  $B$  as  $E$  to  $F$ .

PROOF. Because  $A$  is to  $B$  as  $C$  to  $D$ ,

$m$  times  $A$  and  $m$  times  $C$

are both greater than, both equal to or both less than  
 $n$  times  $B$  and  $n$  times  $D$  respectively,

for all values of  $m$  and  $n$ ; (Def. 5.)

and because  $E$  is to  $F$  as  $C$  to  $D$ ,

$m$  times  $E$  and  $m$  times  $C$

are both greater than, both equal to, or both less than  
 $n$  times  $F$  and  $n$  times  $D$  respectively,

for all values of  $m$  and  $n$ ;

therefore  $m$  times  $A$  and  $m$  times  $E$

are both greater than, both equal to or both less than  
 $n$  times  $B$  and  $n$  times  $F$  respectively,

for all values of  $m$  and  $n$ ;

therefore  $A$  is to  $B$  as  $E$  to  $F$ .

(Def. 5.)

Wherefore, *ratios, which are equal, &c.*

\* Algebraically. If  $a : b = c : d$ , and  $e : f = c : d$ ,  
then  $a : b = e : f$ .

## PROPOSITION 6. [EUCLID ELEM. PROP. 12.]

*If any number of ratios be equal, each ratio is equal to the ratio of the sum of the antecedents to the sum of the consequents\*.*

Let  $A, B, C, D, E, F$  be any number of magnitudes of the same kind, such that the ratios of  $A$  to  $B, C$  to  $D, E$  to  $F$  are equal:

it is required to prove that

$A$  is to  $B$  as the sum of  $A, C, E$  to the sum of  $B, D, F$ .

CONSTRUCTION. Take any equimultiples, say the  $m^{\text{th}}$  multiples, of  $A, C, E$ , and any equimultiples, say the  $n^{\text{th}}$  multiples, of  $B, D, F$ .

PROOF. Because  $A$  is to  $B$  as  $C$  to  $D$ , and also as  $E$  to  $F$ ,  
therefore  $m$  times  $A, m$  times  $C$  and  $m$  times  $E$   
are simultaneously all greater than, all equal to or all less  
than  $n$  times  $B, n$  times  $D$  and  $n$  times  $F$  respectively,  
for all values of  $m$  and  $n$ ; (Def. 5.)

therefore  $m$  times  $A$  and  $m$  times the sum of  $A, C, E$   
are simultaneously all greater than, all equal to or all less  
than  $n$  times  $B$  and  $n$  times the sum of  $B, D, F$  respectively,  
for all values of  $m$  and  $n$ .

Therefore

$A$  is to  $B$  as the sum of  $A, C, E$  to the sum of  $B, D, F$ .  
(Def. 5.)

Wherefore, if any number of ratios, &c.

COROLLARY. *The ratio of two magnitudes is equal to the ratio of any two equimultiples of them†.*

\* Algebraically. If  $a : b = c : d = e : f$ ,  
then  $a : b = a + c + e : b + d + f$ .

† Algebraically.  $a : b = ma : mb$ .

## PROPOSITION 7. [EUCLID ELEM. PROP. 13.]

*If the first of three ratios be equal to the second and the second greater than the third, the first is greater than the third\*.*

Let  $A, B, C, D, E, F$  be six magnitudes, such that the ratio of  $A$  to  $B$  is equal to that of  $C$  to  $D$ , and the ratio of  $C$  to  $D$  is greater than that of  $E$  to  $F$ :

it is required to prove that the ratio of  $A$  to  $B$  is greater than that of  $E$  to  $F$ .

PROOF. Because the ratio of  $C$  to  $D$  is greater than that of  $E$  to  $F$ , it is possible to find some equimultiples, say the  $m^{\text{th}}$  multiples, of  $C$  and  $E$ , and some equimultiples, say the  $n^{\text{th}}$  multiples, of  $D$  and  $F$ , such that

$m$  times  $C$  is greater than  $n$  times  $D$

and  $m$  times  $E$  not greater than  $n$  times  $F$ . (Def. 7.)

Again, because  $A$  is to  $B$  as  $C$  to  $D$ ,

$m$  times  $A$  and  $m$  times  $C$

are simultaneously both greater than, both equal to, or both less than  $n$  times  $B$  and  $n$  times  $D$  respectively,

for all values of  $m$  and  $n$ . (Def. 5.)

Therefore for some values of  $m$  and  $n$

$m$  times  $A$  is greater than  $n$  times  $B$

and  $m$  times  $E$  not greater than  $n$  times  $F$ .

Therefore the ratio of  $A$  to  $B$  is greater than that of  $E$  to  $F$ . (Def. 7.)

Wherefore, if the first of three ratios, &c.

\* Algebraically. If  $a : b = c : d$ , and  $c : d > e : f$ ,  
then  $a : b > e : f$ .

## PROPOSITION 10. [EUCLID ELEM. PROP. 17.]

*If the sum of the first and the second of four magnitudes be to the second as the sum of the third and the fourth to the fourth, the first is to the second as the third to the fourth\*.*

Let  $A, B, C, D$  be four magnitudes,  $A$  and  $B$  being of the same kind and  $C$  and  $D$  of the same kind, such that the sum of  $A$  and  $B$  is to  $B$  as the sum of  $C$  and  $D$  to  $D$ : it is required to prove that

$A$  is to  $B$  as  $C$  to  $D$ .

CONSTRUCTION. Take any equimultiples, say the  $m^{\text{th}}$  multiples, of  $A, B, C, D$  and any equimultiples, say the  $n^{\text{th}}$  multiples, of  $B, D$ ; then the sums of  $m$  times  $B$  and  $n$  times  $B$  and of  $m$  times  $D$  and  $n$  times  $D$  are equimultiples of  $B$  and  $D$  respectively.

PROOF. Because the sum of  $A$  and  $B$  is to  $B$  as the sum of  $C$  and  $D$  to  $D$ , (Hypothesis.)  
therefore  $m$  times the sum of  $A$  and  $B$   
and  $m$  times the sum of  $C$  and  $D$  are simultaneously  
both greater than, both equal to or both less than  
the sum of  $m$  and  $n$  times  $B$   
and the sum of  $m$  and  $n$  times  $D$  respectively,  
for all values of  $m$  and  $n$ : (Def. 5.)  
therefore  $m$  times  $A$  and  $m$  times  $C$   
are simultaneously both greater than, both equal to or both  
less than  $n$  times  $B$  and  $n$  times  $D$  respectively,  
for all values of  $m$  and  $n$ ;  
therefore  $A$  is to  $B$  as  $C$  to  $D$ . (Def. 5.)

Wherefore, *if the sum, &c.*

\* Algebraically. If  $a+b : b = c+d : d$ , then  $a : b = c : d$ .



## PROPOSITION 11. [EUCLID ELEM. PROP. 18.]

*If the first of four magnitudes be to the second as the third to the fourth, then the sum of the first and the second is to the second as the sum of the third and the fourth to the fourth\*.*

Let  $A, B, C, D$  be four magnitudes,  $A$  and  $B$  being of the same kind, and  $C$  and  $D$  of the same kind, such that  $A$  is to  $B$  as  $C$  to  $D$ :

it is required to prove that

the sum of  $A$  and  $B$  is to  $B$  as the sum of  $C$  and  $D$  to  $D$ .

CONSTRUCTION. Take any equimultiples, say the  $m^{\text{th}}$  multiples, of  $A, C$  and any equimultiples, say the  $n^{\text{th}}$  multiples, of  $B, D$ .

PROOF. Because  $A$  is to  $B$  as  $C$  to  $D$ , (Hypothesis.)  
therefore  $m$  times  $A$  and  $m$  times  $C$   
are simultaneously both greater than, both equal to or both  
less than  $n$  times  $B$  and  $n$  times  $D$  respectively,  
for all values of  $m$  and  $n$ ; (Def. 5.)

therefore  $m$  times the sum of  $A$  and  $B$   
and  $m$  times the sum of  $C$  and  $D$   
are simultaneously both greater than, both equal to or both  
less than the sum of  $m$  and  $n$  times  $B$   
and the sum of  $m$  and  $n$  times  $D$  respectively,  
for all values of  $m$  and  $n$ .

And it is manifest that

$m$  times the sum of  $A$  and  $B$   
and  $m$  times the sum of  $C$  and  $D$   
are simultaneously both greater than  
 $p$  times  $B$  and  $p$  times  $D$  respectively,  
for all values of  $m$  and  $p$ , where  $m$  is not less than  $p$ .

Therefore  $m$  times the sum of  $A$  and  $B$   
and  $m$  times the sum of  $C$  and  $D$   
are simultaneously both greater than, both equal to or both  
less than  $p$  times  $B$  and  $p$  times  $D$  respectively,  
for all values of  $m$  and  $p$ ;  
therefore the sum of  $A$  and  $B$  is to  $B$  as the sum of  $C$   
and  $D$  to  $D$ . (Def. 5.)

Wherefore, *if the first, &c.*

\* Algebraically. If  $a : b = c : d$ , then  $a + b : b = c + d : d$ .

## PROPOSITION 12. [EUCLID ELEM. PROP. 19.]

*If the sum of the first and the second of four magnitudes be to the sum of the third and the fourth as the second to the fourth, the first is to the second as the third to the fourth\*.*

Let  $A, B, C, D$  be four magnitudes of the same kind, such that

the sum of  $A$  and  $B$  is to the sum of  $C$  and  $D$  as  $B$  to  $D$  :  
it is required to prove that  $A$  is to  $B$  as  $C$  to  $D$ .

PROOF. Because  
the sum of  $A$  and  $B$  is to the sum of  $C$  and  $D$  as  $B$  to  $D$ ,  
the sum of  $A$  and  $B$  is to  $B$  as the sum of  $C$  and  $D$  to  $D$ .

(Prop. 9.)

Therefore  $A$  is to  $B$  as  $C$  is to  $D$ . (Prop. 10.)

Wherefore, *if the sum, &c.*

\* Algebraically. If  $a+b:c+d=b:d$ , then  $a:b=c:d$ .

## PROPOSITION 13. [EUCLID ELEM. PROP. 20.]

*If the first of six magnitudes be to the second as the fourth to the fifth, and the second be to the third as the fifth to the sixth, then the first and the fourth are both greater than, both equal to, or both less than the third and the sixth respectively\*.*

Let  $A, B, C, D, E, F$  be six magnitudes,  $A, B, C$  being of the same kind and  $D, E, F$  of the same kind, such that

$A$  is to  $B$  as  $D$  to  $E$ , and  $B$  is to  $C$  as  $E$  to  $F$ :

it is required to prove that  $A$  and  $D$  are both greater than, both equal to or both less than  $C$  and  $F$  respectively.

PROOF. First, let  $A$  be greater than  $C$ .

Because  $A$  is to  $B$  as  $D$  to  $E$ , (Hypothesis.)  
and the ratio of  $A$  to  $B$  is greater than that of  $C$  to  $B$ ,

(Prop. 2.)

the ratio of  $D$  to  $E$  is greater than that of  $C$  to  $B$ ; (Prop. 7.)

and because  $C$  is to  $B$  as  $F$  to  $E$ ; (Def. 5, Note.)

the ratio of  $D$  to  $E$  is greater than that of  $F$  to  $E$ ; (Prop. 7.)

therefore  $D$  is greater than  $F$ . (Prop. 4.)

Secondly, because the magnitudes are proportionals when taken in the orders  $A, B, D, E$ ;  $B, C, E, F$ , they are also proportionals when taken in the orders  $D, E, A, B$ ;  $E, F, B, C$ ;

(Def. 5, Note.)

therefore by the first case,

if  $D$  be greater than  $F$ ,  $A$  is greater than  $C$ .

Lastly. The magnitudes are also proportionals when taken in the orders  $C, B, F, E$ ;  $B, A, E, D$ ; (Def. 5, Note.)  
therefore by the first and second cases,

if  $C$  be greater than  $A$ ,  $F$  is greater than  $D$ ,

and if  $F$  be greater than  $D$ ,  $C$  is greater than  $A$ ;

therefore  $A$  and  $D$  are both greater than, both equal to or both less than  $C$  and  $F$  respectively.

Wherefore, *if the first, &c.*

\* Algebraically. If  $a : b = d : e$  and  $b : c = e : f$ ,  
then  $a > < c$  according as  $d > < f$ .

## PROPOSITION 14. [EUCLID ELEM. PROP. 22.]

*If the first of six magnitudes be to the second as the fourth to the fifth, and the second be to the third as the fifth to the sixth, then the first is to the third as the fourth to the sixth\*.*

Let  $A, B, C, D, E, F$  be six magnitudes,  $A, B, C$  being of the same kind and  $D, E, F$  of the same kind, such that  $A$  is to  $B$  as  $D$  to  $E$ , and  $B$  is to  $C$  as  $E$  to  $F$ : it is required to prove that  $A$  is to  $C$  as  $D$  to  $F$ .

CONSTRUCTION. Take any equimultiples, say the  $m^{\text{th}}$  multiples, of  $A, D$ ,  
and any equimultiples, say the  $n^{\text{th}}$  multiples, of  $B, E$ ,  
and any equimultiples, say the  $p^{\text{th}}$  multiples, of  $C, F$ .

PROOF. Because  $A$  is to  $B$  as  $D$  to  $E$ ,  
and  $B$  is to  $C$  as  $E$  to  $F$ ;  
therefore  $m$  times  $A$  is to  $n$  times  $B$  as  $m$  times  $D$  to  $n$  times  $E$ ,  
and  $n$  times  $B$  is to  $p$  times  $C$  as  $n$  times  $E$  to  $p$  times  $F$ .  
(Prop. 1.)

Therefore  $m$  times  $A$  and  $m$  times  $D$   
are both greater than, both equal to or both less than  
 $p$  times  $C$  and  $p$  times  $F$  respectively,  
for all values of  $m$  and  $p$ . (Prop. 13.)  
Therefore  $A$  is to  $C$  as  $D$  to  $F$ . (Def. 5.)

Wherefore, *if the first, &c.*

COROLLARY. *Ratios which are duplicate of equal ratios are equal†.*

\* Algebraically. If  $a : b = d : e$  and  $b : c = e : f$ , then  $a : c = d : f$ .

† Algebraically. If  $a : b = b : c$  and  $d : e = e : f$  and  $a : b = d : e$ ,  
then  $a : c = d : f$ .

## PROPOSITION 15.

*If the first of six magnitudes have to the second a greater ratio than the fourth to the fifth, and the second have to the third a greater ratio than the fifth to the sixth, then the first has to the third a greater ratio than the fourth to the sixth\*.*

Let  $A, B, C, D, E, F$  be six magnitudes,  $A, B, C$  being of the same kind, and  $D, E, F$  of the same kind, such that

$A$  has to  $B$  a greater ratio than  $D$  to  $E$ ,

and  $B$  has to  $C$  a greater ratio than  $E$  to  $F$ ;

it is required to prove that

$A$  has to  $C$  a greater ratio than  $D$  to  $F$ .

CONSTRUCTION. Because the ratio of  $A$  to  $B$  is greater than that of  $D$  to  $E$ , it is possible to find some equimultiples, say the  $m^{\text{th}}$  multiples, of  $A$  and  $D$ , and some equimultiples, say the  $n^{\text{th}}$  multiples, of  $B$  and  $E$ , such that

$m$  times  $A$  is greater than  $n$  times  $B$ ,

and  $m$  times  $D$  not greater than  $n$  times  $E$ : (Def. 7.)

and because the ratio of  $B$  to  $C$  is greater than that of  $E$  to  $F$ , it is possible to find some equimultiples, say the  $p^{\text{th}}$  multiples, of  $B$  and  $E$ , and some equimultiples, say the  $q^{\text{th}}$  multiples, of  $C$  and  $F$ , such that

$p$  times  $B$  is greater than  $q$  times  $C$ ,

and  $p$  times  $E$  not greater than  $q$  times  $F$ . (Def. 7.)

Let  $p$  times  $m$  be  $r$  and  $n$  times  $q$  be  $s$ ,

and let  $n$  times  $B$  be  $H$  and  $p$  times  $E$  be  $K$ .

PROOF. Because  $m$  times  $A$  is greater than  $n$  times  $B$ ,

and  $p$  times  $m$  is  $r$ , and  $n$  times  $B$  is  $H$ ,

therefore  $r$  times  $A$  is greater than  $p$  times  $H$ ;

and because  $p$  times  $B$  is greater than  $q$  times  $C$ ,

and  $p$  times  $B$  is  $K$  and  $n$  times  $q$  is  $s$ ,

therefore  $n$  times  $K$  is greater than  $s$  times  $C$ ;

and because  $n$  times  $B$  is  $H$ , and  $p$  times  $E$  is  $K$ ,

$p$  times  $H$  is equal to  $n$  times  $K$ ;

therefore  $r$  times  $A$  is greater than  $s$  times  $C$ .

Similarly it can be proved that

$r$  times  $D$  is not greater than  $s$  times  $F$ ;

therefore  $A$  has to  $C$  a greater ratio than  $D$  to  $F$ . (Def. 7.)

Wherefore, *if the first, &c.*

\* Algebraically. If  $a : b > d : e$  and  $b : c > e : f$ ,  
then  $a : c > d : f$ .

## PROPOSITION 16.

*Ratios, of which equal ratios are duplicate, are equal\*.*

Let  $A, B, C, D, E, F$  be six magnitudes,  $A, B, C$  being of the same kind, and  $D, E, F$  of the same kind, such that

$A$  is to  $B$  as  $B$  to  $C$ ,

and  $D$  is to  $E$  as  $E$  to  $F$ ,

and also  $A$  is to  $C$  as  $D$  to  $F$ :

it is required to prove that

$A$  is to  $B$  as  $D$  to  $E$ .

PROOF. If the ratio of  $A$  to  $B$  were greater than that of  $D$  to  $E$ ,

then also, since  $A$  is to  $B$  as  $B$  to  $C$ ,

and  $D$  is to  $E$  as  $E$  to  $F$ ,

the ratio of  $B$  to  $C$  would be greater than that of  $E$  to  $F$ ;

(Prop. 7.)

therefore the ratio of  $A$  to  $C$  would be greater than that of  $D$  to  $F$ .

(Prop. 15.)

Similarly, if the ratio of  $A$  to  $B$  were less than  $D$  to  $E$ , then the ratio of  $A$  to  $C$  would be less than that of  $D$  to  $F$ ;

therefore  $A$  is to  $B$  as  $D$  to  $E$ .

Wherefore, *ratios, of which &c.*

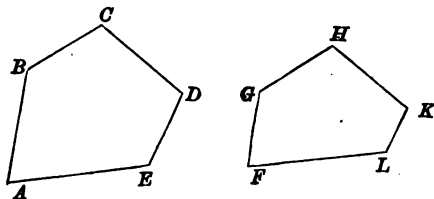
\* Algebraically. If  $a : c = d : f$  and  $a : b = b : c$  and  $d : e = e : f$ ,  
then  $a : b = d : e$ ,  
negative quantities being excluded.

## BOOK VI.

### DEFINITIONS.

It is often convenient to speak of closed rectilineal figures as a class. The wording of definition 15 of Book I. (page 11) implies that the term **polygon** does not include a triangle or a quadrilateral. This restriction for the future will not be maintained, and any closed rectilineal figure, no matter what the number of its sides may be, will be included under the term polygon.

**DEFINITION 1.** *When the angles of one polygon taken in order are equal to the angles of another taken in order, the two polygons are said to be equiangular to one another.*



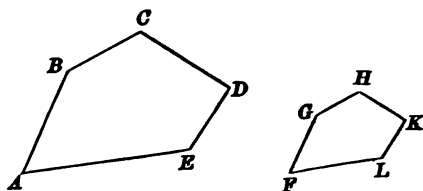
The polygons  $ABCDE$ ,  $FGHLK$  are equiangular to one another, if the angles at  $A$ ,  $B$ ,  $C$ ,  $D$ ,  $E$  be equal to the angles at  $F$ ,  $G$ ,  $H$ ,  $K$ ,  $L$  respectively.

Pairs of vertices  $A$ ,  $F$ ;  $B$ ,  $G$ ; &c., at which the angles are equal, are **corresponding vertices**: and pairs of sides  $AB$ ,  $FG$ ;  $BC$ ,  $GH$ ; &c. joining corresponding pairs of vertices are **corresponding sides**.

In this definition there is one more condition of equality than is necessary. If  $n-1$  of the angles of a polygon of  $n$  sides be equal to  $n-1$  of the angles of another polygon of  $n$  sides, the remaining angles must be equal. (See I. Prop. 32, Coroll.)

**DEFINITION 2.** *When the ratio of a side of one of two polygons, which are equiangular to one another, to the corresponding side of the other is the same for all pairs of corresponding sides, the polygons are said to be similar to one another.*

The polygons  $ABCDE$ ,  $FGHKL$  are similar to one another, if the angles at  $A, B, C, D, E$  be equal to the angles at  $F, G, H, K, L$



respectively, and if also all the ratios of  $AB$  to  $FG$ ,  $BC$  to  $GH$ ,  $CD$  to  $HK$ ,  $DE$  to  $KL$ ,  $EA$  to  $LF$  be equal to one another.

It will be seen hereafter that there are in this definition three more conditions of equality than are necessary. One unnecessary condition is contained in the statement that the polygons are equiangular to one another. Other two unnecessary conditions are contained in the statement that *all* the ratios are equal. For it can be proved that in general, if in two *equiangular* polygons all but two of the ratios of corresponding sides be equal, all the ratios are equal.

**DEFINITION 3.** *If in each of two given finite straight lines a point be taken such that the segments of the first line are in the same ratio as the segments of the second line, the two lines are said to be cut proportionally by the points.*



In the diagram (figure 1) the points  $P, Q$  cut the straight lines  $AB, CD$  proportionally, if  $AP$  be to  $PB$  as  $CQ$  to  $QD$ .

This definition is extended also to the case, where the points  $P$  and  $Q$  are in the lines  $AB, CD$  produced (figure 2). It must however be noticed that both points  $P, Q$  must be in the lines themselves, or both points in the lines produced: otherwise the lines are not said to be cut proportionally.

In figure 1 the points  $P, Q$  are said to cut the lines  $AB, CD$  internally; in figure 2 the points  $P, Q$  are said to cut the lines  $AB, CD$  externally.



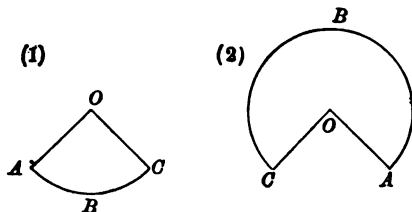
**DEFINITION 4.** *In some cases, where one side of a triangle is specially distinguished from the other two sides, that side is called the **base** of the triangle and the perpendicular\* upon that side from the opposite vertex is called the **altitude** of the triangle.*

Similarly, one side of a parallelogram is sometimes called the **base** and the perpendicular distance between it and the opposite side the **altitude** of the parallelogram.

**DEFINITION 5.** *When a straight line is divided into two parts, so that the whole is to one part as that part to the other part, the line is said to be divided in **extreme and mean ratio**.*

**DEFINITION 6.** *The figure formed of an arc of a circle and the radii drawn to its extremities is called a **sector** of the circle.*

*The angle between the radii, which is subtended by the arc, is called the **angle of the sector**.*



If  $O$  be the centre of the circle, of which the arc  $ABC$  is a part, and  $OA$ ,  $OC$  be radii, the figure  $OABC$  is a sector.

In figure 1 the angle of the sector is less than two right angles, in figure 2 the angle of the sector is greater than two right angles.

**DEFINITION 7.** *Points lying on a straight line are said to be **collinear**. A set of such points is called a **range**.*

*Straight lines passing through a point are said to be **concurrent**. A set of such lines is called a **pencil**. The lines are called the **rays** of the pencil and the point is called the **vertex** of the pencil.*

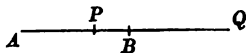
A set of points  $ABCD\dots$  lying on a straight line is called the **range**  $ABCD\dots$

\* (See I. Def. 11, p. 9.)

A set of straight lines drawn from a point  $O$  to a series of points  $A, B, C, D, \dots$  is called the pencil  $O$  ( $ABCD, \dots$ ).

**DEFINITION 8.** *Four points on a straight line, such that one pair divide the straight line joining the other pair internally and externally in the same ratio, are called a harmonic range.*

In the diagram, if  $AP$  be to  $PB$  as  $AQ$  to  $QB$ , then  $A, P, B, Q$  is a harmonic range.



Because  $AP$  is to  $PB$  as  $AQ$  to  $QB$ , therefore  $QB$  is to  $BP$  as  $QA$  to  $AP$  (V. Def. 5, Note, and V. Prop. 9), that is, the points  $B, A$  divide the distance  $QP$  internally and externally in the same ratio.

*The two points which form either pair of a harmonic range are said to be conjugate to one another.*

The points  $A, B$  and  $P, Q$  are two pairs of conjugate points.

**DEFINITION 9.** *A pencil of four rays passing through the four points of a harmonic range is called a harmonic pencil.*

*Two rays, which pass through a pair of conjugate points of a harmonic range, are called conjugate rays of the pencil.*

**DEFINITION 10.** *Four points on a straight line, such that one pair divide the straight line joining the other pair internally and externally in different ratios, are called an anharmonic range.*

*The ratio of the ratio of internal division to the ratio of external division is called the ratio of the anharmonic range.*

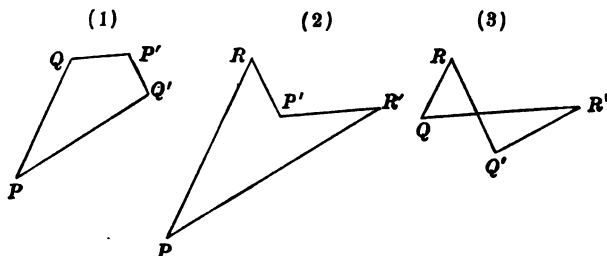
*If two anharmonic ranges have equal ratios, they are called like anharmonic ranges.*

**DEFINITION 11.** *A pencil of four rays passing through the four points of an anharmonic range is called an anharmonic pencil.*

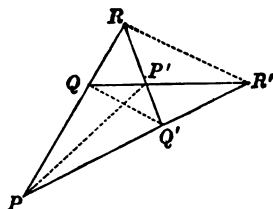
*The ratio of the range is also said to be the ratio of the pencil.*

*If two pencils have equal ratios, they are called like an-harmonic pencils.*

Quadrilaterals are often divided into three classes (1) convex, (2) re-entrant, (3) cross, the natures of which appear from the adjoining diagrams.



If the sides of a quadrilateral be produced both ways, the character of the complete figure which is so formed is independent of the class to which the quadrilateral belongs, as is evident from the adjoining diagram.



Such a figure is called a complete quadrilateral, of which the following is a definition.

**DEFINITION 12.** *The figure formed by four infinite straight lines is called a complete quadrilateral.*

*The straight line joining the intersection of one pair of lines to the intersection of the other pair is called a diagonal.*

There are three such diagonals. In the last diagram  $PP'$ ,  $QQ'$ ,  $RR'$  are diagonals of the complete quadrilateral there represented.

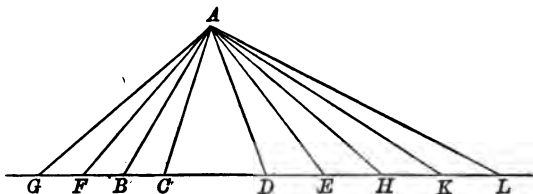
## PROPOSITION 1.

*The ratio of two triangles of the same altitude is equal to the ratio of their bases.*

Let the triangles  $ABC$ ,  $ADE$  be two triangles of the same altitude, that is, having a common vertex  $A$  and their bases  $BC$ ,  $DE$  in a straight line:

it is required to prove that the triangle  $ABC$  is to the triangle  $ADE$  as  $BC$  to  $DE$ .

CONSTRUCTION. In  $CB$  produced, take any number of straight lines  $BF$ ,  $FG$  each equal to  $BC$ ; and in  $DE$  produced, any number  $EH$ ,  $HK$ ,  $KL$  each equal to  $DE$ ; and draw  $AF$ ,  $AG$ ,  $AH$ ,  $AK$ ,  $AL$ .



PROOF. Because  $BC$ ,  $FB$ ,  $GF$  are equal, the triangles  $ABC$ ,  $AFB$ ,  $AGF$  are equal. (I. Prop. 38, Coroll.) Therefore the triangle  $AGC$  is the same multiple of the triangle  $ABC$ , that  $GC$  is of  $BC$ .

Similarly it can be proved that the triangle  $ADL$  is the same multiple of the triangle  $ADE$ , that  $DL$  is of  $DE$ .

Again, if  $GC$  be equal to  $DL$ ,

the triangle  $AGC$  is equal to the triangle  $ADL$ ,

(I. Prop. 38, Coroll.)

and if  $GC$  be greater or less than  $DL$ , the triangle  $AGC$  is greater or less respectively than the triangle  $ADL$ .

Therefore since of the four magnitudes the triangles  $ABC$ ,  $ADE$  and the lines  $BC$ ,  $DE$ , the triangle  $AGC$  and the line  $GC$  are any equimultiples whatever of the first and

the two triangles are congruent  
 equilateral triangles  
 it has been proved that  
 the triangles are congruent  
 hence the two triangles are  
 congruent

therefore

the triangles are congruent

Wherefore the two triangles are congruent

Corollary - The two triangles are congruent  
 hence the two triangles are congruent

Corollary - The two triangles are congruent  
 hence the two triangles are congruent

Corollary - The two triangles are congruent  
 hence the two triangles are congruent

Each parallelogram is a rectangle  
 hence the two triangles are congruent  
 hence the two triangles are congruent  
 hence the two triangles are congruent

# THEOREM

1. The diagonals of a parallelogram bisect each other  
 parallel and one of them bisects the other at the point of intersection

2. The two triangles formed by the diagonals of a parallelogram  
 triangles are congruent

3. If a line segment connects the midpoints of two sides of a triangle  
 triangle  $ABC$  and all produced to  $BC$  in  $D$ , the triangles  $ABC$   
 $ADC$  have the same area as  $ABC$

4. If in the same triangle  $ABC$  a line segment  $DE$  is drawn such  
 that  $BE$  is twice  $ED$  and  $CE$  twice  $ED$  are the segments  $BE$  and  
 $CE$  respectively  $1/3$  of the base  $BC$ , the triangles  $ABC$  and  $ADE$   
 $ABC$  are in the ratio of the numbers  $1$  to  $9$

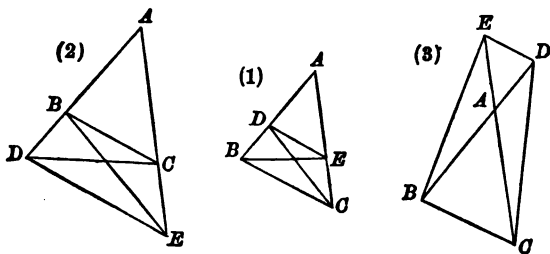
## PROPOSITION 2. PART 1.

*If a straight line be parallel to one side of a triangle, it cuts the other sides proportionally.*

Let the straight line  $DE$  be parallel to the side  $BC$  of the triangle  $ABC$ , and cut the sides  $AB$ ,  $AC$  or these sides produced in  $D$ ,  $E$  respectively:

it is required to prove that  $BD$  is to  $DA$  as  $CE$  to  $EA$ .

CONSTRUCTION. Draw  $BE$ ,  $CD$ .



PROOF. Because the two triangles  $BDE$ ,  $CDE$  have the side  $DE$  common, and  $BC$  is parallel to  $DE$ ,

the triangles  $BDE$ ,  $CDE$  are equal. (I. Prop. 37.)

Therefore the triangle  $BDE$  is to the triangle  $ADE$  as the triangle  $CDE$  to the triangle  $ADE$ . (V. Def. 5, Note.)

But the triangle  $BDE$  is to the triangle  $ADE$  as  $BD$  to  $DA$ ; (Prop. 1.)

and the triangle  $CDE$  is to the triangle  $ADE$  as  $CE$  to  $EA$ ; (Prop. 1.)

therefore  $BD$  is to  $DA$  as  $CE$  to  $EA$ . (V. Prop. 5.)

Wherefore, *if a straight line &c.*

COROLLARY. Because  $BC$  is parallel to  $DE$  a side of the triangle  $ADE$ , it follows that

$DB$  is to  $BA$  as  $EC$  to  $CA$ ;

therefore also  $AB$  is to  $AD$  as  $AC$  to  $AE$ .

(V. Props. 10, and 11.)

CONSTRUCTIONS

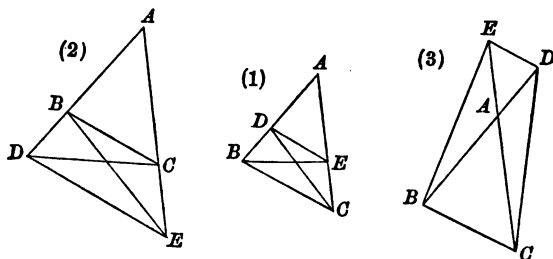
1. A line is drawn through the vertex  $A$  of a triangle  $ABC$  parallel to the base  $BC$ . If  $P$  is a point on this line, then the line segment  $AP$  is equal to the line segment  $BC$ .
2. If a line is drawn through the vertex  $A$  of a triangle  $ABC$  parallel to the base  $BC$ , and if  $P$  is a point on this line, then the line segment  $AP$  is equal to the line segment  $BC$ .
3. Through a point  $P$  on a line  $AB$ , a line is drawn parallel to the line  $AC$ . If  $Q$  is a point on this line, then the line segment  $PQ$  is equal to the line segment  $AC$ .
4. If a line is drawn through the vertex  $A$  of a triangle  $ABC$  parallel to the base  $BC$ , and if  $P$  is a point on this line, then the line segment  $AP$  is equal to the line segment  $BC$ .
5. If a line is drawn through the vertex  $A$  of a triangle  $ABC$  parallel to the base  $BC$ , and if  $P$  is a point on this line, then the line segment  $AP$  is equal to the line segment  $BC$ .
6.  $ABC$  is a triangle. A line is drawn through the vertex  $A$  parallel to the base  $BC$ . A line is drawn through the vertex  $B$  parallel to the side  $AC$ . These two lines intersect at the point  $D$ . The line segment  $AD$  is equal to the line segment  $BC$ .
7. Through a given point  $P$ , a line is drawn parallel to a given line  $AB$ . This line intersects the line  $AC$  at the point  $Q$ . The line segment  $PQ$  is equal to the line segment  $AC$ .
8. Find a point  $P$  on a line  $AB$  such that the line segment  $AP$  is equal to the line segment  $BC$ .

## PROPOSITION 2. PART 2.

*If a straight line cut two sides of a triangle proportionally, it is parallel to the third side.*

Let the straight line  $DE$  cut the sides  $AB$ ,  $AC$  of the triangle  $ABC$ , or these sides produced, proportionally in  $D$ ,  $E$  respectively, so that  $BD$  is to  $DA$  as  $CE$  to  $EA$ :  
it is required to prove that  $DE$  is parallel to  $BC$ .

CONSTRUCTION. Draw  $BE$ ,  $CD$ .



PROOF. Because  $BD$  is to  $DA$  as  $CE$  to  $EA$ ,  
(Hypothesis.)  
and as  $BD$  to  $DA$  so is the triangle  $BDE$  to the triangle  $ADE$ ,  
(Prop. 1.)  
and as  $CE$  to  $EA$  so is the triangle  $CDE$  to the triangle  $ADE$ ;  
(Prop. 1.)  
therefore the triangle  $BDE$  is to the triangle  $ADE$   
as the triangle  $CDE$  to the triangle  $ADE$ ; (V. Prop. 5.)  
i.e. the triangles  $BDE$ ,  $CDE$  have the same ratio to the triangle  $ADE$ ;  
therefore the triangles  $BDE$ ,  $CDE$  are equal.  
(V. Prop. 3.)  
But these triangles have a common side  $DE$  and lie on the same side of it;  
therefore  $BC$  is parallel to  $DE$ . (I. Prop. 39.)

Wherefore, *if a straight line &c.*



COROLLARY. If  $AB$  be to  $AD$  as  $AC$  to  $AE$ ,  
it follows that  $BD$  is to  $DA$  as  $CE$  to  $EA$   
(V. Props. 10 and 11.)  
and therefore  $BC$  is parallel to  $AD$ .

## EXERCISES.

1. Prove that there is only one point which divides a given straight line internally in a given ratio, and only one point which divides a given straight line externally in a given ratio.

2. If  $DEF$  be a triangle inscribed in a triangle  $ABC$  and have its sides parallel to those of  $ABC$ , then  $D$ ,  $E$ ,  $F$  must be the middle points of  $BC$ ,  $CA$ ,  $AB$ .

3. From a point  $E$  in the common base of two triangles  $ACB$ ,  $ADB$ , straight lines are drawn parallel to  $AC$ ,  $AD$ , meeting  $BC$ ,  $BD$  at  $F$ ,  $G$ : shew that  $FG$  is parallel to  $CD$ .

4. If two given distances  $PQ$ ,  $RS$  be measured off on two fixed parallel straight lines, then the locus of the intersection of each of the pairs  $PS$ ,  $QR$  and  $PR$ ,  $QS$  is a parallel straight line.

5. On three straight lines  $OAP$ ,  $OBQ$ ,  $OCR$  the points are chosen so that  $AB$ ,  $PQ$  are parallel and  $BC$ ,  $QR$  are parallel; prove that  $AC$ ,  $PR$  also are parallel.

6. If two opposite sides  $AB$ ,  $DC$  of a quadrilateral  $ABCD$  be parallel, any straight line  $PQ$  which cuts  $AD$ ,  $BC$  proportionally must be parallel to  $AB$  and  $DC$ .

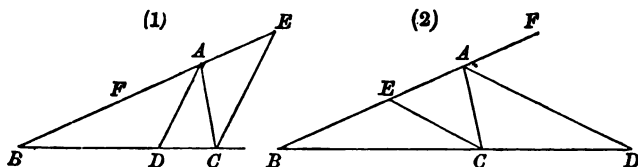
7. Take  $D$ ,  $E$ , the middle points of the sides  $CA$ ,  $CB$  of a triangle; join  $D$  and  $E$ , and draw  $AE$ ,  $BD$ , intersecting in  $O$ ; then the areas of the triangles  $DOE$ ,  $EOB$ ,  $BOA$  are in continued proportion.

## PROPOSITION 3. PART 1.

*If an angle of a triangle be bisected internally or externally by a straight line which cuts the opposite side or that side produced, the ratio of the segments of that side is equal to the ratio of the other sides of the triangle.*

Let the angle  $BAC$  of the triangle  $ABC$  be bisected internally or externally by the straight line  $AD$  which cuts in  $D$  the opposite side  $BC$  (fig. 1) or  $BC$  produced (fig. 2): it is required to prove that  $BD$  is to  $DC$  as  $BA$  to  $AC$ .

CONSTRUCTION. Through  $C$  draw  $CE$  parallel to  $DA$  to meet  $BA$  produced or  $BA$  in  $E$ ; and take in  $BA$  or  $BA$  produced a point  $F$  on the side of  $A$  away from  $E$ .



PROOF. Because  $AC$  intersects the parallels  $AD$ ,  $EC$ , the angle  $DAC$  is equal to the angle  $ACE$ ; (I. Prop. 29.) and because  $FAE$  intersects the parallels  $AD$ ,  $EC$ , the angle  $FAD$  is equal to the angle  $AEC$ . (I. Prop. 29.) And the angle  $DAC$  is equal to the angle  $FAD$ ;

(Hypothesis.)  
therefore the angle  $AEC$  is equal to the angle  $ACE$ ;  
therefore  $AC$  is equal to  $AE$ . (I. Prop. 6.)

Now because  $AD$  is parallel to  $EC$ , one of the sides of the triangle  $BEC$ ,

$BD$  is to  $DC$  as  $BA$  to  $AE$ ; (Prop. 2.)  
and  $AC$  has been proved equal to  $AE$ ;  
therefore  $BD$  is to  $DC$  as  $BA$  to  $AC$ .

Wherefore, if an angle &c.

## EXERCISES.

1.  $ABC$  is a triangle which has its base  $BC$  bisected in  $D$ .  $DE$ ,  $DF$  bisect the angles  $ADC$ ,  $ADB$  meeting  $AC$ ,  $AB$  in  $E$ ,  $F$ . Prove that  $EF$  is parallel to  $BC$ .

2. If  $AD$  bisect the angle  $BAC$ , and meet  $BC$  in  $D$ , and  $DE$ ,  $DF$  bisect the angles  $ADC$ ,  $ADB$  and meet  $AC$ ,  $AB$  in  $E$ ,  $F$  respectively, then the triangle  $BEF$  is to the triangle  $CEF$  as  $BA$  is to  $AC$ .

3. An internal point  $O$  is joined to the vertices of a triangle  $ABC$ . The bisectors of the angles  $BOC$ ,  $COA$ ,  $AOB$  meet  $BC$ ,  $CA$ ,  $AB$  respectively in  $D$ ,  $E$ ,  $F$ : prove that the ratio compounded of the ratios  $AE$  to  $EC$ ,  $CD$  to  $DB$ , and  $BF$  to  $FA$  is unity.

4. One circle touches another internally at  $O$ . A straight line touches the inner circle at  $C$ , and meets the outer one in  $A$ ,  $B$ : prove that  $OA$  is to  $OB$  as  $AC$  to  $CB$ .

5. The angle  $A$  of a triangle  $ABC$  is bisected by  $AD$  which cuts the base at  $D$ , and  $O$  is the middle point of  $BC$ : shew that  $OD$  has the same ratio to  $OB$  that the difference of the sides has to their sum.

6.  $AD$  and  $AE$  bisect the interior and the exterior angles at  $A$  of a triangle  $ABC$ , and meet the base at  $D$  and  $E$ ; and  $O$  is the middle point of  $BC$ : shew that  $OB$  is a mean proportional between  $OD$  and  $OE$ .

7. If  $A$ ,  $B$ ,  $C$  be three points in a straight line, and  $D$  a point at which  $AB$  and  $BC$  subtend equal angles, then the locus of  $D$  is a circle.

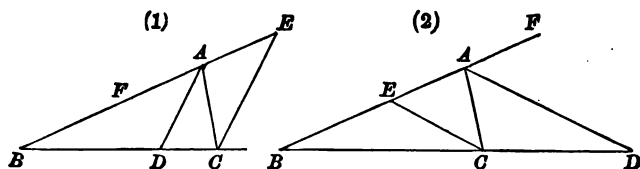
## PROPOSITION 3. PART 2.

*If a straight line drawn through a vertex of a triangle cut the opposite side or that side produced, so that the ratio of the segments of that side is equal to the ratio of the other sides of the triangle, the straight line bisects the vertical angle internally or externally.*

Let the straight line  $AD$  drawn through  $A$  one of the vertices of the triangle  $ABC$  cut the opposite side  $BC$  or  $BC$  produced in  $D$ , so that  $BD$  is to  $DC$  as  $BA$  to  $AC$ :

it is required to prove that  $AD$  bisects the angle at  $A$  internally or externally.

CONSTRUCTION. Through  $C$  draw  $CE$  parallel to  $DA$  to meet  $BA$  produced (fig. 1) or  $BA$  (fig. 2) in  $E$ ; and take in  $BA$  or  $BA$  produced a point  $F$  on the side of  $A$  away from  $E$ .



PROOF. Because  $DA$  is parallel to  $CE$  one of the sides of the triangle  $BEC$ ,

$BD$  is to  $DC$  as  $BA$  to  $AE$ . (Prop. 2.)

And  $BD$  is to  $DC$  as  $BA$  to  $AC$ ; (Hypothesis.)

therefore  $BA$  is to  $AC$  as  $BA$  to  $AE$ ; (V. Prop. 5.)

therefore  $AE$  is equal to  $AC$ ; (V. Prop. 3.)

therefore the angle  $ACE$  is equal to the angle  $AEC$ .

(I. Prop. 5.)

Again, because  $AC$  intersects the parallels  $AD$ ,  $EC$ , the angle  $DAC$  is equal to the angle  $ACE$ ; (I. Prop. 29.)

and because  $FAE$  intersects the parallels  $AD$ ,  $EC$ , the angle  $FAD$  is equal to the angle  $AEC$ ; (I. Prop. 29.)

therefore the angle  $DAC$  is equal to the angle  $FAD$ .

Wherefore, *if a straight line &c.*

## EXERCISES.

1. The bisector of the angle  $BAC$  of a triangle  $ABC$  meets  $BC$  in  $D$ ; a straight line  $EGF$  parallel to  $BC$  meets  $AB, AD, AC$  in  $E, G, F$  respectively; prove that  $EG$  is to  $GF$  as  $BD$  to  $DC$ .

2. The sides  $AB, AC$  of a given triangle  $ABC$  are produced to any points  $D, E$ , so that  $DE$  is parallel to  $BC$ . The straight line  $DE$  is divided at  $F$  so that  $DF$  is to  $FE$  as  $BD$  is to  $CE$ : shew that the locus of  $F$  is a straight line.

3. If a chord of a circle  $AB$  be divided at  $C$  so that  $AC$  is to  $CB$  as  $AP$  to  $PB$ , where  $P$  is a point on the circle: then a circle can be described to touch  $AB$  at  $C$  and the given circle at  $P$ .

4.  $ABCD$  is a quadrilateral: if the bisectors of the angles at  $A$  and  $C$  meet in  $BD$ , then the bisectors of the angles at  $B$  and  $D$  meet in  $AC$ .

5. If  $A, B, C, D$  be four points in order on a straight line, such that  $AB$  is to  $BC$  as  $AD$  to  $DC$ , and  $P$  be any point on the circle described on  $BD$  as diameter, then  $PB, PD$  are the bisectors of the angle  $APC$ .

## PROPOSITION 4.

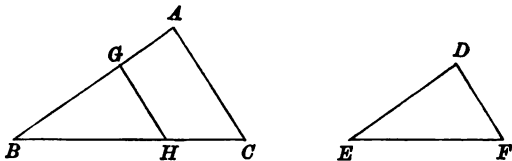
*If two triangles be equiangular to one another, they are similar.*

Let the triangles  $ABC$ ,  $DEF$  be equiangular to one another:

it is required to prove that the ratios

$AB$  to  $DE$ ,  $BC$  to  $EF$  and  $CA$  to  $FD$  are equal.

CONSTRUCTION. Of the two lines  $BA$ ,  $ED$  let  $BA$  be the greater\*. In  $BA$  take  $BG$  equal to  $ED$ , and in  $BC$  take  $BH$  equal to  $EF$ ; and draw  $GH$ .



PROOF. Because in the triangles  $GBH$ ,  $DEF$ ,  
 $BG$  is equal to  $ED$ , and  $BH$  to  $EF$ ,  
 and the angle  $GBH$  to the angle  $DEF$ ,  
 the triangles are equal in all respects; (I. Prop. 4.)  
 therefore the angle  $BGH$  is equal to the angle  $EDF$ ;  
 and the angle  $EDF$  is equal to the angle  $BAC$ ;  
 (Hypothesis.)

Hence the angle  $BGH$  is equal to the angle  $BAC$ ,  
 and  $AC$  is parallel to  $GH$ ; (I. Prop. 28.)  
 therefore  $BA$  is to  $BG$  as  $BC$  to  $BH$ ;  
 (Prop. 2, Part 1, Coroll.)  
 and  $GB$  is equal to  $DE$ , and  $BH$  to  $EF$ ;  
 therefore  $AB$  is to  $DE$  as  $BC$  to  $EF$ .

Similarly it can be proved that either of these ratios is equal to the ratio  $CA$  to  $FD$ .

Wherefore, if two triangles &c.

\* The case when  $BA$  is equal to  $ED$  has already been dealt with. (I. Prop. 26.)

## EXERCISES.

1. A common tangent to two circles cuts the straight line joining the centres externally or internally in the ratio of the radii.

2. If  $AB$ ,  $CD$ , two parallel straight lines, be divided proportionally by  $P$ ,  $Q$ , so that  $AP$  is to  $PB$  as  $CQ$  to  $QD$ , then  $AC$ ,  $PQ$ ,  $BD$  meet in a point.

3.  $ABCD$  is a parallelogram;  $P$  and  $Q$  are points in a straight line parallel to  $AB$ ;  $PA$  and  $QB$  meet at  $R$ , and  $PD$  and  $QC$  meet at  $S$ ; shew that  $RS$  is parallel to  $AD$ .

4. The tangents at the points  $P$  and  $Q$  of a circle intersect in  $T$ : if from any other point  $R$  of the circle the perpendiculars  $RM$ ,  $RN$  be drawn to the tangents  $TP$  and  $TQ$ , and the perpendicular  $RL$  be drawn to the chord  $PQ$ , then  $RL$  is a mean proportional between  $RM$  and  $RN$ .

5. A straight line, parallel to the side  $BC$  of a triangle  $ABC$ , meets the sides  $AB$ ,  $AC$  (or those sides produced) at  $D$  and  $E$ . On  $DE$  is constructed a parallelogram  $DEFG$ , and the straight lines  $BG$ ,  $CF$  (produced if necessary) meet each other at  $S$ . Prove that  $AS$  is parallel to  $DG$  or  $EF$ .

6. Inscribe an equilateral triangle in a given triangle, so as to have one side parallel to a side of the given triangle.

7. If two triangles have their bases equal and in the same straight line, and also have their vertices on a parallel straight line, any straight line parallel to their bases will cut off equal areas from the two triangles.

8. In a given triangle  $ABC$  draw a straight line  $PQ$  parallel to  $AB$  meeting  $AC$ ,  $BC$  in  $P$ ,  $Q$ , so that  $PQ$  may be a mean proportional between  $BQ$ ,  $QC$ .

9. Two circles intersect at  $A$ , and a straight line is drawn bisecting the angle between the tangents at  $A$ . Prove that the segments of the line cut off by the circles are proportional to the radii.

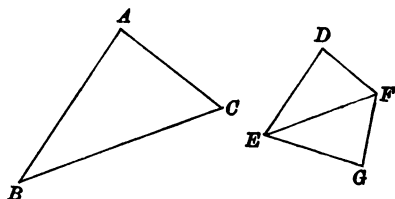
10. If  $ACB$ ,  $BCD$  be equal angles, and  $DB$  be perpendicular to  $BC$  and  $BA$  to  $AC$ , then the triangle  $DBC$  is to the triangle  $ABC$  as  $DC$  is to  $CA$ .

## PROPOSITION 5.

*If the ratios of the three sides of one triangle to the three sides of another triangle be equal, the triangles are equiangular to one another.*

Let the given triangles  $ABC$ ,  $DEF$  be such that the ratios  $AB$  to  $DE$ ,  $BC$  to  $EF$  and  $CA$  to  $FD$  are equal: it is required to prove that the triangles  $ABC$ ,  $DEF$  are equiangular to one another.

CONSTRUCTION. On the side of  $EF$  away from  $D$  draw  $EG$ ,  $FG$  making the angles  $FEG$ ,  $EFG$  equal to the angles  $CBA$ ,  $BCA$  respectively. (I. Prop. 23.)



PROOF. Because in the triangles  $ABC$ ,  $GEF$ , the angles  $ABC$ ,  $BCA$  are equal to the angles  $GEF$ ,  $EFG$ , the triangles  $ABC$ ,  $GEF$  are equiangular to one another, (I. Prop. 32.)

and therefore  $AB$  is to  $GE$  as  $BC$  to  $EF$ . (Prop. 4.)

And  $AB$  is to  $DE$  as  $BC$  to  $EF$ ; (Hypothesis.)

therefore  $AB$  is to  $GE$  as  $AB$  to  $DE$ ; (V. Prop. 5.)

therefore  $GE$  is equal to  $DE$ . (V. Prop. 3.)

Similarly it can be proved that  $GF$  is equal to  $DF$ .

Then because in the triangles  $DEF$ ,  $GEF$ ,

$DE$ ,  $EF$ ,  $FD$  are equal to  $GE$ ,  $EF$ ,  $FG$  respectively,

the triangles are equal in all respects; (I. Prop. 8.)

therefore the triangles  $DEF$ ,  $GEF$  are equiangular to one another;

and the triangle  $GEF$  was constructed so as to be equiangular to the triangle  $ABC$ ;

therefore the triangles  $ABC$ ,  $DEF$  are equiangular to one another.

Wherefore, if the ratios &c.



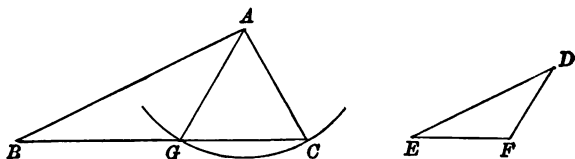
## PROPOSITION 5. A.

*If one pair of angles of two triangles be equal and another pair of angles be supplementary, the ratios of the sides opposite to these pairs of angles are equal.*

Let  $ABC$ ,  $DEF$  be two triangles in which the angles  $ABC$ ,  $DEF$  are equal, and the angles  $ACB$ ,  $DFE$  are supplementary :

it is required to prove that  $AB$  is to  $DE$  as  $AC$  to  $DF$ .

CONSTRUCTION. Of the two angles  $ACB$ ,  $DFE$ , let  $ACB$  be the less. With  $A$  as centre and  $AC$  as radius describe a circle cutting  $BC$  in  $G$ ; and draw  $AG$ .



PROOF. Because  $AC$  is equal to  $AG$ ,  
the angle  $AGC$  is equal to the angle  $ACG$ ; (I. Prop. 5.)  
and the angle  $AGB$  is the supplement of the angle  $AGC$ ,  
and the angle  $DFE$  is the supplement of the angle  $ACB$ ;  
(Hypothesis.)

therefore the angle  $AGB$  is equal to the angle  $DFE$ ;  
and the angle  $ABG$  is equal to the angle  $DEF$ ;  
(Hypothesis.)

therefore the triangles  $ABG$ ,  $DEF$  are equiangular to one another; (I. Prop. 32.)

therefore  $AB$  is to  $DE$  as  $AG$  to  $DF$ ; (Prop. 4.)

and  $AC$  is equal to  $AG$ ; (Constr.)

therefore  $AB$  is to  $DE$  as  $AC$  to  $DF$ .

Wherefore, *if one pair of angles, &c.*

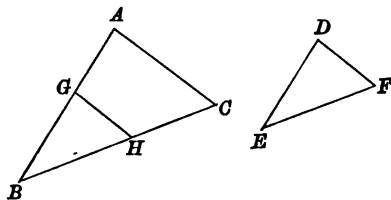
## PROPOSITION 6.

*If the ratios of two sides of one triangle to two sides of another triangle be equal, and also the angles contained by those sides be equal, the triangles are equiangular to one another.*

Let  $ABC$ ,  $DEF$  be two triangles in which  $AB$  is to  $DE$  as  $BC$  to  $EF$ ,

and the angle  $ABC$  is equal to the angle  $DEF$ :  
it is required to prove that the triangles  $ABC$ ,  $DEF$  are equiangular to one another.

CONSTRUCTION. Of the two lines  $BA$ ,  $ED$  let  $BA$  be the greater\*. In  $BA$  take  $BG$  equal to  $ED$ , and in  $BC$  take  $BH$  equal to  $EF$ ; and draw  $GH$ .



PROOF. Because  $BA$  is to  $ED$  as  $BC$  to  $EF$ ,  
and  $BG$  is equal to  $ED$ , and  $BH$  to  $EF$ ,  
therefore  $BA$  is to  $BG$  as  $BC$  to  $BH$ ;  
therefore  $GH$  is parallel to  $AC$ , (Prop. 2, Part 2, Coroll.)  
and the angle  $BGH$  is equal to the angle  $BAC$ .  
(I. Prop. 29.)

Again, because in the triangles  $DEF$ ,  $GBH$ ,  
 $ED$  is equal to  $BG$  and  $EF$  to  $BH$ ,  
and the angle  $DEF$  to the angle  $GBH$ ,  
the triangles are equal in all respects; (I. Prop. 4.)

\* The case when  $BA$  is equal to  $ED$  has already been dealt with.  
(I. Prop. 4.)

therefore the angle  $EDF$  is equal to the angle  $BGH$ ,  
and therefore to the angle  $BAC$ .

And the angles at  $B$  and  $E$  are equal ; (Hypothesis.)  
therefore the triangles  $ABC$ ,  $DEF$  are equiangular to one  
another. (I. Prop. 32.)

Wherefore, *if the ratios &c.*

## EXERCISES.

1. Shew that the locus of the middle points of straight lines parallel to the base of a triangle and terminated by its sides is a straight line.

2.  $CAB$ ,  $CEB$  are two triangles having the angle  $B$  common and the sides  $CA$ ,  $CE$  equal ; if  $BAE$  be produced to  $D$  and  $ED$  be taken a third proportional to  $BA$ ,  $AC$ , then the triangle  $BDC$  is similar to the triangle  $BAC$ .

3. From a point  $E$  in the common base of the triangles  $ACB$ ,  $ADB$ , straight lines are drawn parallel to  $AC$ ,  $AD$ , meeting  $BC$ ,  $BD$  in  $F$  and  $G$  ; shew that  $FG$ ,  $CD$  are parallel.

4.  $C$  is a point in a given straight line  $AB$ , and  $AB$  is produced to  $O$ , so that  $CO$  is a mean proportional between  $AO$  and  $BO$ . If  $P$  be any point on a circle described with centre  $O$  and radius  $OC$ , then the angles  $APC$ ,  $BPC$  are equal.

5. If a point  $O$  be taken within a parallelogram  $ABCD$ , such that the angles  $OBA$ ,  $ODA$  are equal, then the angles  $OAD$ ,  $OCD$  are equal.

6. If two points  $P$ ,  $Q$  be such that when four perpendiculars  $PM$ ,  $Pm$ ,  $QN$ ,  $Qn$  are dropped upon the straight lines  $AMN$ ,  $Amm$ ,  $PM$  is to  $Pm$  as  $QN$  to  $Qn$ , then  $P$  and  $Q$  lie on a straight line through  $A$ .

7. If on the three sides of any triangle, equilateral triangles be described either all externally or all internally, the centres of the circles inscribed in these triangles are the vertices of an equilateral triangle.

8. The straight line  $OP$  joining a fixed point  $O$  to a variable point  $P$  on a fixed circle is divided in  $Q$  in a constant ratio ; prove that the locus of  $Q$  is a circle.

9. Given the base and the vertical angle of a triangle, find the locus of the intersection of bisectors of sides.

## PROPOSITION 7.

*If the ratios of two sides of one triangle to two sides of another triangle be equal, and also the angles opposite to one pair of these sides be equal, the angles opposite to the other pair of sides are equal or supplementary.*

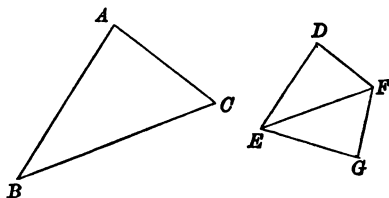
Let  $ABC$ ,  $DEF$  be two triangles, in which

$AB$  is to  $DE$  as  $BC$  to  $EF$ ,

and the angle  $BAC$  is equal to the angle  $EDF$ :

it is required to prove that the angles  $ACB$ ,  $DFE$  are either equal or supplementary.

CONSTRUCTION. On the side of  $EF$  away from  $D$ , draw  $EG$  making the angle  $FEG$  equal to the angle  $CBA$ , and draw  $FG$  making the angle  $EFG$  equal to the angle  $BCA$ . (I. Prop. 23.)



PROOF. Because the triangles  $ABC$ ,  $GEF$  are equiangular to one another, (I. Prop. 32.)

$AB$  is to  $GE$  as  $AC$  to  $GF$ ; (Prop. 4.)

and  $AB$  is to  $DE$  as  $BC$  to  $EF$ ; (Hypothesis.)

therefore  $AB$  is to  $GE$  as  $AB$  to  $DE$ , (V. Prop. 5.)

and  $GE$  is equal to  $DE$ . (V. Prop. 3.)

Now because in the triangles  $GEF$ ,  $DEF$ ,

$GE$  is equal to  $DE$  and  $EF$  to  $EF$ ,

and the angle  $EGF$  to the angle  $EDF$ ;

(for each is equal to the angle at  $A$ )

therefore the angles  $GFE$ ,  $DFE$  are either equal or supplementary; (I. Prop. 26, A.)

and the angle  $GFE$  is equal to the angle  $ACB$ ; (Constr.)

therefore the angles  $ACB$ ,  $DFE$  are either equal or supplementary.

Wherefore, if the ratios &c.

**COROLLARY.** *When two of the ratios of a side of one triangle to the corresponding side of another triangle are equal, and also the angles opposite to one pair of these sides equal, the triangles are equiangular to one another, provided that of the angles opposite to the second pair of sides,*

- (1) *each be less than a right angle,*
- (2) *each be greater than a right angle,*
- or (3) *one of them be a right angle.*

(I. Prop. 26 A, Coroll.)

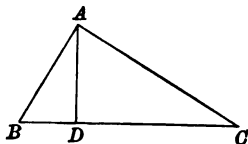
### EXERCISE.

Prove that, if  $ABCD$ ,  $EFGH$  be two quadrilaterals, such that the angles  $ABC$ ,  $ADC$  are equal to the angles  $EFG$ ,  $EHG$  respectively, and the ratios  $AB$  to  $EF$ ,  $BC$  to  $FG$ ,  $CD$  to  $GH$  are equal, and if the angles  $BAD$ ,  $FEH$  be both acute angles, then the quadrilaterals are similar.

## PROPOSITION 8.

*In a right-angled triangle, if a perpendicular be drawn from the opposite vertex to the hypotenuse, the perpendicular is a mean proportional between the segments of the hypotenuse, and each of the sides of the triangle is a mean proportional between the hypotenuse and the segment of it adjacent to that side.*

Let  $ABC$  be a right-angled triangle, and let  $AD$  be drawn perpendicular to the hypotenuse  $BC$ : it is required to prove that  $BD$  is to  $DA$  as  $AD$  to  $DC$ , that  $BC$  is to  $BA$  as  $BA$  to  $BD$ , and that  $BC$  is to  $CA$  as  $CA$  to  $CD$ .



**PROOF.** Because in the triangles  $ABC$ ,  $DBA$ , the right angle  $BAC$  is equal to the right angle  $BDA$ , and the angle  $ABC$  is equal to the angle  $DBA$ , therefore the triangles  $ABC$ ,  $DBA$  are equiangular to one another. (I. Prop. 32.)

Similarly it can be proved that the triangles  $DAC$ ,  $ABC$  are equiangular to one another.

Therefore the triangles  $DBA$ ,  $DAC$  are equiangular to one another.

Now, because the triangles  $DBA$ ,  $DAC$  are equiangular to one another,

$BD$  is to  $DA$  as  $AD$  to  $DC$ ; (Prop. 4.)

and because the triangles  $ABC$ ,  $DBA$  are equiangular to one another,

$BC$  is to  $BA$  as  $BA$  to  $BD$ ; (Prop. 4.)

and because the triangles  $ABC$ ,  $DAC$  are equiangular to one another,

$BC$  is to  $CA$  as  $AC$  to  $CD$ . (Prop. 4.)

Wherefore, in a right-angled triangle &c.

## EXERCISES.

1. If the perpendicular drawn from the vertex of a triangle to the base be a mean proportional between the segments of the base, the triangle is right-angled.

2. If a triangle whose sides are unequal can be divided into two similar triangles by a straight line joining the vertex to a point in the base, the vertical angle must be a right angle.

3. If  $CD$ ,  $CE$ , the internal and the external bisectors of the angle at  $C$  in a triangle  $ABC$  having a right angle at  $A$ , cut  $BA$  in  $D$  and  $E$  respectively, then  $AC$  is a mean proportional between  $AD$ ,  $AE$ .

4. A perpendicular  $AD$  is drawn to the hypotenuse  $BC$  of a right-angled triangle from the opposite vertex  $A$ : and perpendiculars  $DE$ ,  $DF$  are drawn from  $D$  to the sides  $AB$ ,  $AC$ ; prove that a circle will pass through the four points  $B$ ,  $E$ ,  $F$ ,  $C$ .

5. On the tangent to a circle at  $A$  two points  $C$  and  $B$  are taken such that  $AC$  is equal to  $CB$ : the straight lines joining  $B$ ,  $C$  to  $F$ , the opposite extremity of the diameter through  $A$ , cut the circle in  $D$ ,  $E$  respectively; prove that  $AE$  is to  $ED$  as  $FA$  to  $FD$ .

6. A chord  $CD$  is drawn parallel to a diameter  $AB$  of a circle, and  $AC$ ,  $AD$  are produced to cut the tangent at  $B$  in  $E$ ,  $F$  respectively; prove that the sum of the rectangles  $AC$ ,  $CE$  and  $AD$ ,  $DF$  is equal to the square on  $AB$ .

7. If  $A$  be a point outside a circle and  $B$  be the middle point of the chord of contact of tangents drawn from  $A$ , and  $P$ ,  $Q$  be any two points on the circle, then  $PA$  is to  $QA$  as  $PB$  to  $QB$ .

8. Two circles intersect in  $A$ ,  $B$ ; from  $B$  perpendiculars  $BE$ ,  $BF$  are drawn to their diameters  $AC$ ,  $AD$ ; prove that  $C$ ,  $E$ ,  $F$ ,  $D$  lie on a circle, which is cut at right angles by the circle whose centre is  $A$  and radius  $AB$ .

9. The circumference of one circle passes through the centre of another circle. If from any point of the former circle two straight lines be drawn to touch the latter circle, the straight line joining the points of contact is bisected by the common chord of the two circles.

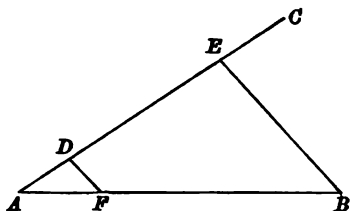
## PROPOSITION 9.

*From a given finite straight line to cut off any aliquot part required.*

Let  $AB$  be the given finite straight line:  
it is required to cut off from  $AB$  a given aliquot part, say the  $n^{\text{th}}$  part.

CONSTRUCTION. From  $A$  draw any straight line  $AC$  making an angle with  $AB$ , and in it take any point  $D$ , and cut off  $AE$  the same multiple of  $AD$  that  $AB$  is of the part to be cut off, i.e. take  $AE$  equal to  $n$  times  $AD$ .

Draw  $EB$ , and draw  $DF$  parallel to it meeting  $AB$  in  $F$ :  
then  $AF$  is the part required.



PROOF. Because  $FD$  is parallel to  $BE$ , one of the sides of the triangle  $ABE$ ,

$AB$  is to  $AF$  as  $AE$  to  $AD$ ; (Prop. 2, Part 1, Coroll.)

and  $AE$  is equal to  $n$  times  $AD$ ;

therefore  $AB$  is equal to  $n$  times  $AF$ .

Therefore  $AF$  is the  $n^{\text{th}}$  part of  $AB$ .

Wherefore, from the given straight line  $AB$ ,  $AF$  the part required has been cut off.



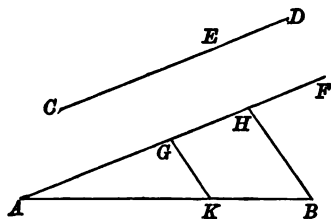
## PROPOSITION 10.

*To divide a given finite straight line similarly to a given divided straight line.*

Let  $AB$  be a given straight line and  $CD$  another given straight line divided in  $E$ :  
it is required to divide  $AB$  similarly to  $CD$ .

CONSTRUCTION. Draw  $AF$  making an angle with  $AB$ ; cut off  $AG, GH$  equal to  $CE, ED$  respectively.

Draw  $HB$ , and draw  $GK$  parallel to  $HB$  meeting  $AB$  in  $K$ :  
then  $AB$  is divided at  $K$  similarly to  $CD$  at  $E$ .



PROOF. Because  $GK$  is parallel to  $HB$  one of the sides of the triangle  $AHB$ ,

$AK$  is to  $KB$  as  $AG$  to  $GH$ ; (Prop. 2.)  
and  $AG$  is equal to  $CE$ , and  $GH$  to  $ED$ . (Constr.)  
Therefore  $AK$  is to  $KB$  as  $CE$  to  $ED$ .

Wherefore, the straight line  $AB$  has been divided at  $K$  similarly to the straight line  $CD$  at  $E$ .

## EXERCISES.

1. If three straight lines passing through a point  $O$  cut two parallel straight lines  $ABC, PQR$  in  $A, P; B, Q; C, R$ , then the lines  $AC, PR$  are similarly divided in  $B, Q$ .
2. Draw a straight line through a given point  $A$ , so that the perpendiculars upon it from two other given points  $B$  and  $C$  may be in a given ratio.
3. Draw through two given points on a circle two parallel chords which shall have a given ratio to one another.

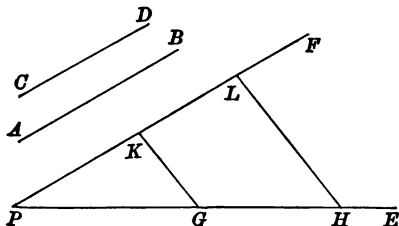
## PROPOSITION 11.

*To find a third proportional to two given finite straight lines.*

Let  $AB, CD$  be two given straight lines:  
it is required to find a third proportional to  $AB, CD$ .

CONSTRUCTION. Draw from any point  $P$  a pair of straight lines  $PE, PF$  making an angle with one another, and from  $PE$  cut off  $PG, GH$  equal to  $AB, CD$  respectively and from  $PF$  cut off  $PK$  equal to  $CD$ .

Draw  $GK$  and draw  $HL$  parallel to  $GK$  meeting  $PF$  in  $L$ :  
then  $KL$  is a third proportional to  $AB, CD$ .



PROOF. Because  $GK$  is parallel to  $HL$  one of the sides of the triangle  $PHL$ ,

$PG$  is to  $GH$  as  $PK$  to  $KL$ ; (Prop. 2.)  
and  $PG$  is equal to  $AB$  and  $GH$  and  $PK$  are each equal to  $CD$ ;

therefore  $AB$  is to  $CD$  as  $CD$  to  $KL$ .

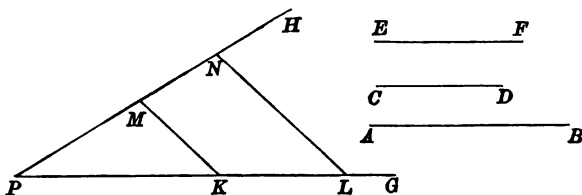
Wherefore to the two given straight lines  $AB, CD$  a third proportional  $KL$  has been found.

## PROPOSITION 12.

*To find a fourth proportional to three given straight lines.*

Let  $AB$ ,  $CD$ ,  $EF$  be three given straight lines:  
it is required to find a fourth proportional to  $AB$ ,  $CD$ ,  $EF$ .

CONSTRUCTION. Draw from any point  $P$  a pair of straight lines  $PG$ ,  $PH$  making an angle with one another; and from  $PG$  cut off  $PK$ ,  $KL$  equal to  $AB$ ,  $CD$  respectively, and from  $PH$  cut off  $PM$  equal to  $EF$ . Draw  $KM$ , and draw  $LN$  parallel to  $KM$  meeting  $PH$  in  $N$ : then  $MN$  is a fourth proportional to  $AB$ ,  $CD$ ,  $EF$ .



PROOF. Because  $KM$  is parallel to  $LN$  one of the sides of the triangle  $PLN$ ,

$PK$  is to  $KL$  as  $PM$  to  $MN$ ; (Prop. 2.)  
and  $PK$  is equal to  $AB$ ,  $KL$  to  $CD$ , and  $PM$  to  $EF$ ;  
therefore  $AB$  is to  $CD$  as  $EF$  to  $MN$ .

Wherefore to the three given straight lines  $AB$ ,  $CD$ ,  $EF$ , a fourth proportional  $MN$  has been found.

## EXERCISE.

1.  $C$  is a point on a straight line  $AB$ ; find a point  $D$  in  $AB$  produced, such that  $DA$  is to  $DB$  as  $CA$  to  $CB$ .

## PROPOSITION 13.

*To find a mean proportional between two given straight lines.*

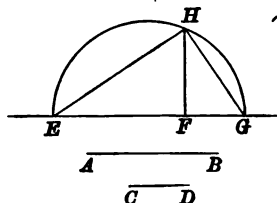
Let  $AB$ ,  $CD$  be two given straight lines:  
it is required to find a mean proportional between  $AB$  and  $CD$ .

CONSTRUCTION. Draw any straight line and from it cut off  $EF$ ,  $FG$  equal to  $AB$ ,  $CD$  respectively.

Describe a circle on  $EG$  as diameter and draw  $FH$  at right angles to  $EG$  meeting the circle in  $H$ :

then  $FH$  is a mean proportional between  $AB$  and  $CD$ .

Draw  $EH$ ,  $HG$ .



PROOF. Because  $EHG$  is a semicircle,  
the angle  $EHG$  is a right angle; (III. Prop. 31.)  
and because  $HF$  is the perpendicular from  $H$  on the hypotenuse of the right-angled triangle  $EHG$ ,  
 $EF$  is to  $FH$  as  $FH$  to  $FG$ ; (Prop. 8.)  
and  $EF$  is equal to  $AB$  and  $FG$  to  $CD$ ;  
therefore  $AB$  is to  $FH$  as  $FH$  to  $CD$ .

Wherefore, between the two given straight lines  $AB$ ,  $CD$   
a mean proportional  $FH$  has been found.

## EXERCISES.

1. Find a mean proportional between two given straight lines by the use of the theorem of Proposition 37 of Book III.
2. Divide a given finite straight line into two parts, so that their mean proportional may be of given length.
3. Construct an isosceles triangle equal to a given triangle and having the vertical angle equal to one of the angles of the given triangle.
4. Find a third proportional to two given straight lines by a method similar to that of Proposition 13.

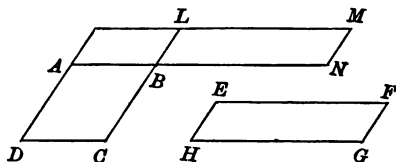
**DEFINITION.** *If the ratio of a side of one polygon to a side of another polygon be equal to the ratio of an adjacent side of the second to an adjacent side of the first, those sides are said to be reciprocally proportional.*

**PROPOSITION 14. PART 1.**

*If two parallelograms, which have a pair of equal angles, be equal in area, their sides about the equal angles are reciprocally proportional.*

Let  $ABCD$ ,  $EFGH$  be two parallelograms, which have the angles at  $B$  and  $H$  equal, and which are equal in area: it is required to prove that  $AB$  is to  $HG$  as  $EH$  to  $BC$ .

**CONSTRUCTION.** From  $AB$ ,  $CB$  produced cut off  $BN$ ,  $BL$  equal to  $HG$ ,  $EH$ , and complete the parallelograms  $AL$ ,  $LN$ .



**PROOF.** Because in the parallelograms  $LN$ ,  $EG$ ,  
 $LB$  is equal to  $EH$ , and  $BN$  to  $HG$ ,  
 and the angle  $LBN$  to the angle  $EHG$ ,  
 therefore the parallelograms  $LN$ ,  $EG$  are equal in area.  
 (I. Props. 4 and 34.)

And the area of  $EG$  is equal to the area of  $AC$ ;  
 therefore the area of  $AL$  is to the area of  $LN$  as the area  
 of  $AL$  to the area of  $AC$ .

And  $AB$  is to  $BN$  as the area of  $AL$  to the area of  $LN$ ,  
 and  $LB$  is to  $BC$  as the area of  $AL$  to the area of  $AC$ ;

(Prop. 1, Coroll. 3.)

therefore  $AB$  is to  $BN$  as  $LB$  to  $BC$ , (V. Prop. 5.)

that is,  $AB$  is to  $HG$  as  $EH$  to  $BC$ .

Wherefore, if two parallelograms &c.

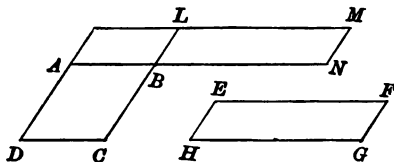
## PROPOSITION 14. PART 2.

*If two parallelograms, which have a pair of equal angles, have their sides about the equal angles reciprocally proportional, the parallelograms are equal in area.*

Let  $ABCD$ ,  $EFGH$  be two parallelograms, which have the angles at  $B$ ,  $H$  equal and in which  $AB$  is to  $HG$  as  $EH$  to  $BC$ :

it is required to prove that the parallelograms  $ABCD$ ,  $EFGH$  are equal in area.

CONSTRUCTION. From  $AB$ ,  $CB$  produced cut off  $BN$ ,  $BL$  equal to  $HG$ ,  $EH$  and complete the parallelograms  $AL$ ,  $LN$ .



PROOF. Because in the parallelograms  $LN$ ,  $EG$ ,  
 $LB$  is equal to  $EH$ , and  $BN$  to  $HG$ ,  
 and the angle  $LBN$  to the angle  $EHG$ ,  
 therefore the parallelograms  $LN$ ,  $EG$  are equal in area.  
 (I. Props. 4 and 34.)

Because  $AB$  is to  $HG$  as  $EH$  to  $BC$ ,  
 and  $BN$  is equal to  $HG$  and  $LB$  to  $EH$ ,  
 therefore  $AB$  is to  $BN$  as  $LB$  to  $BC$ .

And  $AB$  is to  $BN$  as the area of  $AL$  to the area of  $LN$ ,  
 and  $LB$  is to  $BC$  as the area of  $AL$  to the area of  $AC$ ;  
 (Prop. 1, Coroll. 3.)

therefore the area of  $AL$  is to the area of  $LN$  as the area  
 of  $AL$  to the area of  $AC$ . (V. Prop. 5.)

Therefore the area of  $LN$  is equal to the area of  $AC$ .  
 And the parallelograms  $LN$ ,  $EG$  are equal in area;  
 therefore the parallelograms  $AC$ ,  $EG$  are equal in area.

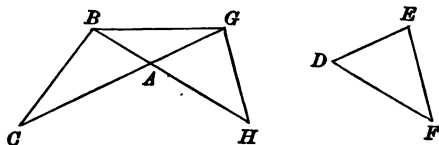
Wherefore, if two parallelograms &c.

## PROPOSITION 15. PART 1.

*If two triangles, which have a pair of equal angles, be equal in area, their sides about the equal angles are reciprocally proportional.*

Let  $ABC$ ,  $DEF$  be two triangles, which have the angles at  $A$ ,  $D$  equal and which are equal in area:  
it is required to prove that  $BA$  is to  $DF$  as  $ED$  to  $AC$ .

CONSTRUCTION. From  $BA$ ,  $CA$  produced cut off  $AH$ ,  $AG$  equal to  $DF$ ,  $DE$  respectively, and draw  $BG$ ,  $GH$ .



PROOF. Because in the triangles  $AGH$ ,  $DEF$ ,  
 $GA$  is equal to  $ED$ , and  $AH$  to  $DF$ ,  
and the angle  $GAH$  to the angle  $EDF$ ,  
the triangles are equal in all respects. (I. Prop. 4.)

And the area of  $DEF$  is equal to the area of  $ABC$ ;  
therefore the area of  $AGH$  is to the area of  $ABC$  as the  
area of  $ABG$  to the area of  $ABC$ .

And  $BA$  is to  $AH$  as the area of  $ABG$  to the area of  $GAH$ ,  
and  $GA$  is to  $AC$  as the area of  $ABG$  to the area of  $ABC$ ;  
(Prop. 1.)

therefore  $BA$  is to  $AH$  as  $GA$  to  $AC$ ; (V. Prop. 5.)  
and  $AH$  is equal to  $DF$  and  $GA$  to  $ED$ ; (Constr.)  
therefore  $BA$  is to  $DF$  as  $ED$  to  $AC$ .

Wherefore, if two triangles &c.

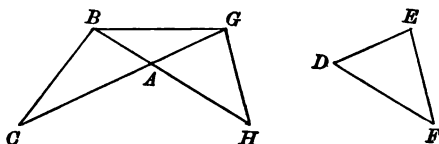


## PROPOSITION 15. PART 2.

*If two triangles, which have a pair of equal angles, have their sides about the equal angles reciprocally proportional, the triangles are equal in area.*

Let  $ABC$ ,  $DEF$  be two triangles, which have the angles at  $A$ ,  $D$  equal and in which  $BA$  is to  $DF$  as  $ED$  to  $AC$ : it is required to prove that the triangles  $ABC$ ,  $DEF$  are equal in area.

CONSTRUCTION. From  $BA$ ,  $CA$  produced cut off  $AH$ ,  $AG$  equal to  $DF$ ,  $DE$  respectively, and draw  $BG$ ,  $GH$ .



PROOF. Because in the triangles  $AGH$ ,  $DEF$ ,  
 $GA$  is equal to  $ED$ , and  $AH$  to  $DF$ ,  
 and the angle  $GAH$  to the angle  $EDF$ ,  
 the triangles are equal in all respects. (I. Prop. 4.)

Because  $BA$  is to  $DF$  as  $ED$  to  $AC$ ,  
 and  $AH$  is equal to  $DF$  and  $GA$  to  $ED$ ;  
 therefore  $BA$  is to  $AH$  as  $GA$  to  $AC$ .

And  $BA$  is to  $AH$  as the area of  $GBA$  to the area of  $AGH$ ,  
 and  $GA$  is to  $AC$  as the area of  $GBA$  to the area of  $ABC$ ;

(Prop. 1.)

therefore the area of  $GBA$  is to the area of  $AGH$  as the  
 area of  $GBA$  to the area of  $ABC$ ;

(V. Prop. 5.)

therefore the area of  $AGH$  is equal to the area of  $ABC$ ;  
 and the triangles  $AGH$ ,  $DEF$ , being equal in all respects,  
 are equal in area;

therefore the triangles  $ABC$ ,  $DEF$  are equal in area.

Wherefore, *if two triangles &c.*

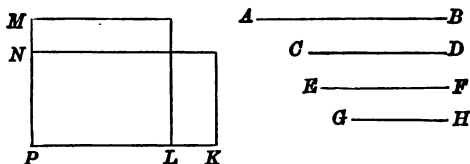
## PROPOSITION 16. PART 1.

*If four straight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means.*

Let the straight lines  $AB, CD, EF, GH$  be proportionals, so that  $AB$  is to  $CD$  as  $EF$  to  $GH$ :

it is required to prove that the rectangle contained by  $AB$  and  $GH$  is equal to the rectangle contained by  $CD$  and  $EF$ .

CONSTRUCTION. From any point  $P$  draw two straight lines at right angles and on one cut off  $PK, PL$  equal to  $AB, CD$  respectively; and on the other cut off  $PM, PN$  equal to  $EF, GH$  respectively; and complete the rectangles  $ML, NK$ .



PROOF. Because  $AB$  is to  $CD$  as  $EF$  to  $GH$ , and  $PK, PL, PM, PN$ , are equal to  $AB, CD, EF, GH$  respectively,

therefore  $PK$  is to  $PL$  as  $PM$  to  $PN$ .

And the angle at  $P$  is common to the two rectangles  $NK, ML$ ;

therefore the rectangle  $NK$  is equal to the rectangle  $ML$ ;

(Prop. 14, Part 2.)

and  $NK$  is contained by  $PK, PN$  and  $ML$  is contained by  $PL, PM$ .

Therefore the rectangle contained by  $AB, GH$  is equal to the rectangle contained by  $CD, EF$ .

Wherefore, if four straight lines &c.

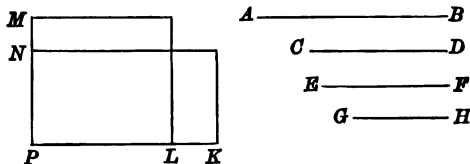
## PROPOSITION 16. PART 2.

*If the rectangle contained by the first and the fourth of four given straight lines be equal to the rectangle contained by the second and the third, the four lines are proportionals.*

Let  $AB, CD, EF, GH$  be four straight lines, such that the rectangle contained by  $AB$  and  $GH$  is equal to the rectangle contained by  $CD$  and  $EF$ :

it is required to prove that  $AB$  is to  $CD$  as  $EF$  to  $GH$ .

CONSTRUCTION. From any point  $P$  draw two straight lines at right angles, and on one cut off  $PK, PL$  equal to  $AB, CD$  respectively; and on the other cut off  $PM, PN$  equal to  $EF, GH$  respectively; and complete the rectangles  $ML, NK$ .



PROOF. Because the rectangle contained by  $AB$  and  $GH$  is equal to the rectangle contained by  $CD$  and  $EF$ , and  $PK, PL, PM, PN$  are equal to  $AB, CD, EF, GH$  respectively,

the rectangle contained by  $PK$  and  $PN$  is equal to the rectangle contained by  $PL$  and  $PM$ ,

that is, the rectangle  $NK$  is equal to the rectangle  $ML$ ;

therefore  $PK$  is to  $PL$  as  $PM$  to  $PN$ ;

(Prop. 14, Part 1.)

and  $PK$  is equal to  $AB$ ,  $PL$  to  $CD$ ,  $PM$  to  $EF$  and  $PN$  to  $GH$ ;

(Constr.)

therefore  $AB$  is to  $CD$  as  $EF$  to  $GH$ .

Wherefore, if the rectangle &c.



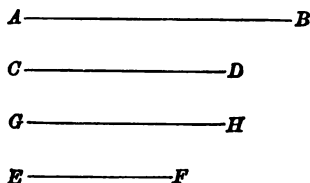
## PROPOSITION 17. PART 2.

*If the rectangle contained by the first and the third of three given straight lines be equal to the square on the second, the three straight lines are proportionals.*

Let  $AB$ ,  $CD$ ,  $EF$  be three given straight lines, such that the rectangle contained by  $AB$  and  $EF$  is equal to the square on  $CD$ :

it is required to prove that  $AB$  is to  $CD$  as  $CD$  to  $EF$ .

CONSTRUCTION. Draw a straight line  $GH$  equal to  $CD$ .



PROOF. Because  $GH$  is equal to  $CD$ , the square on  $CD$  is equal to the rectangle contained by  $CD$  and  $GH$ ;

therefore the rectangle contained by  $AB$  and  $EF$ , which is equal to the square on  $CD$ , is equal to the rectangle contained by  $CD$  and  $GH$ ;

therefore  $AB$  is to  $CD$  as  $GH$  to  $EF$ ; (Prop. 16, Part 2.)

and  $GH$  is equal to  $CD$ ;

therefore  $AB$  is to  $CD$  as  $CD$  to  $EF$ .

Wherefore, *if the rectangle &c.*

## PROPOSITION 18.

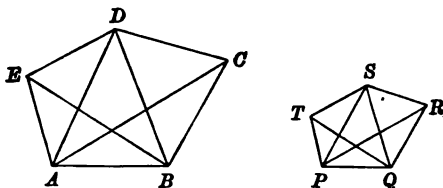
*On a given straight line to construct a polygon similar to a given polygon so that the given straight line may correspond to a given side of the given polygon.*

Let  $AB$  be the given straight line, and  $PQRST$  the given polygon:

it is required to construct on  $AB$  a polygon similar to  $PQRST$  so that  $AB, PQ$  may be corresponding sides.

CONSTRUCTION. From  $P, Q$  the extremities of  $PQ$ , draw the diagonals  $PR, PS, QS, QT$  of the polygon  $PQRST$ . At  $A, B$  on the same side of  $AB$  make the angles  $BAC, BAD, BAE$  equal to the angles  $QPR, QPS, QPT$  respectively, and the angles  $ABC, ABD, ABE$  to the angles  $PQR, PQS, PQT$  respectively, and draw  $CD, DE$ :

then  $ABCDE$  is a polygon constructed as required.



PROOF. Because the triangles  $ABC, PQR$  are equiangular to one another, (Constr.)

$AC$  is to  $PR$  as  $AB$  to  $PQ$ ; (Prop. 4.)

and because the triangles  $ABD, PQS$  are equiangular to one another, (Constr.)

$AD$  is to  $PS$  as  $AB$  to  $PQ$ ; (Prop. 4.)

therefore  $AD$  is to  $PS$  as  $AC$  to  $PR$ . (V. Prop. 5.)

Again, because in the triangles  $DAC, SPR$ ,

the ratios of  $AD$  to  $PS$  and  $AC$  to  $PR$  are equal,

and the angles  $DAC, SPR$  are equal; (Constr.)

therefore the triangles  $DAC, SPR$  are equiangular to one another, and  $CD$  is to  $RS$  as  $AC$  to  $PR$ ; (Prop. 6.)

therefore  $CD$  is to  $RS$  as  $AB$  to  $PQ$ . (V. Prop. 5.)

Again because the triangles  $ABC$ ,  $PQR$  are equiangular to one another, (Constr.)

the angle  $ACB$  is equal to the angle  $PRQ$ ;  
and because the triangles  $DAC$ ,  $SPR$  have been proved equiangular to one another,

the angle  $ACD$  is equal to the angle  $PRS$ ;  
therefore the angle  $BCD$  is equal to the angle  $QRS$ .

Similarly it can be proved that the ratio of any other corresponding pair of sides of the polygons  $ABCDE$ ,  $PQRST$  is equal to that of  $AB$  to  $PQ$ , and that any corresponding pair of angles are equal.

Wherefore, *on the given straight line  $AB$ , the polygon  $ABCDE$  has been constructed similar to the given polygon  $PQRST$ , so that  $AB$ ,  $PQ$  are corresponding sides.*

### EXERCISES.

1. Given the length of the line joining the middle point of a side of a square with an end of the opposite side; determine, by any method, the length of a diagonal of the square.

2. Inscribe in a given triangle a second triangle so that its sides may be parallel to three given straight lines.

In how many ways can this be done?

3. In a triangle  $ABC$  inscribe a square so that two of its vertices may be on  $BC$  and the other two on  $AB$ ,  $AC$ .

4. In a semicircle inscribe a square, so that two corners may lie in the diameter and two on the circumference.

5. In a given sector of a circle inscribe a square so that two corners may lie on the arc and one on each of the bounding radii.

6. In a given sector inscribe a square so that two corners may be on one of the bounding radii, one on the other bounding radius and one on the arc.

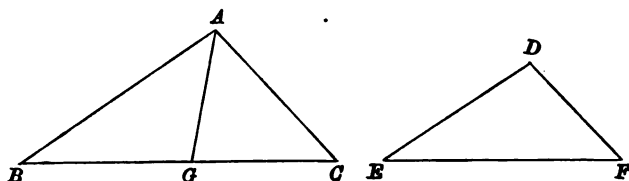
## PROPOSITION 19.

*Similar triangles are to one another in the ratio duplicate of the ratio of two corresponding sides.*

Let  $ABC$ ,  $DEF$  be similar triangles and  $BC$ ,  $EF$  be corresponding sides:

it is required to prove that the triangle  $ABC$  is to the triangle  $DEF$ , in the ratio duplicate of the ratio of  $BC$  to  $EF$ .

CONSTRUCTION. Find a third proportional to  $BC$ ,  $EF$  and from  $BC$  cut off  $BG$  equal to it. Draw  $AG$ .



PROOF. Because the triangles  $ABC$ ,  $DEF$  are similar,  
 $AB$  is to  $DE$  as  $BC$  to  $EF$ ; (Prop. 4.)  
 and  $BC$  is to  $EF$  as  $EF$  to  $BG$ ; (Constr.)  
 therefore  $AB$  is to  $DE$  as  $EF$  to  $BG$ ; (V. Prop. 5.)  
 and the angle  $ABG$  is equal to the angle  $DEF$ ;  
 therefore the triangles  $ABG$ ,  $DEF$  are equal in area.  
 (Prop. 15, Part 2.)

And the triangle  $ABC$  is to the triangle  $ABG$  as  $BC$  to  $BG$ ;  
 therefore the triangle  $ABC$  is to the triangle  $DEF$  as  $BC$   
 to  $BG$ ;

And because  $BC$  is to  $EF$  as  $EF$  to  $BG$ ,  
 $BC$  has to  $BG$  the ratio duplicate of the ratio of  $BC$  to  $EF$ .  
 (V. Def. 9.)

Therefore the triangle  $ABC$  has to the triangle  $DEF$  the  
 ratio duplicate of the ratio of  $BC$  to  $EF$ .

Wherefore, *similar triangles &c.*

COROLLARY. *If three straight lines be proportionals, the first is to the third as any triangle on the first to a similar triangle on the second.*



## EXERCISES.

1. Through a point within a triangle three straight lines are drawn parallel to the sides, dividing the triangle into three triangles and three parallelograms: if the three triangles be equal to each other in area, each is one-ninth of the original triangle.

2. An isosceles triangle is described having each of the angles at the base double of the third angle: if the angles at the base be bisected, and the points where the lines bisecting them meet the opposite sides be joined, the triangle will be divided into two parts having the same ratio as the base to the side of the triangle.

3.  $ABC$  is a triangle, the angle  $A$  being greater than the angle  $B$ : a point  $D$  is taken in  $BC$ , such that the angle  $CAD$  is equal to  $B$ . Prove that  $CD$  is to  $CB$  in the ratio duplicate of the ratio of  $AD$  to  $AB$ .

4. The sides  $BC$ ,  $CA$ ,  $AB$  of an equilateral triangle  $ABC$  are divided in the points  $D$ ,  $E$ ,  $F$  so that the ratios  $BD$  to  $DC$ ,  $CE$  to  $EA$  and  $AF$  to  $FB$  are each equal to 2 to 1. Find the ratio of the triangle  $DEF$  to the triangle  $ABC$ .

5. If a straight line  $AB$  be produced to a point  $C$  so that  $AB$  is a mean proportional between  $AC$  and  $CB$ , then the square on  $AB$  is to the square on  $BC$  as  $AB$  to the excess of  $AB$  over  $BC$ .

6. Find a mean proportional between the areas of two similar right-angled triangles which have one of the sides containing the right angle common.

7. Bisect a given triangle by a line parallel to its base.

8. Bisect a given triangle by a line drawn perpendicular to its base.

9. Divide a given triangle into two parts, having a given ratio to one another, by a straight line parallel to one of its sides.

10.  $ABC$  is a triangle;  $AB$  is produced to  $E$ :  $AD$  is a straight line meeting  $BC$  in  $D$ :  $BF$  is parallel to  $ED$  and meets  $AD$  in  $F$ : construct a triangle similar to  $ABC$  and equal to  $AEF$ .

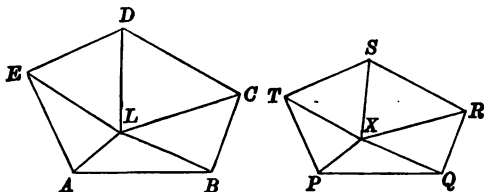
## PROPOSITION 20.

*A pair of similar polygons may be divided into pairs of similar triangles, each pair having the same ratio as the polygons.*

Let  $ABCDE$ ,  $PQRST$  be a pair of similar polygons:  
it is required to prove that the polygons can be divided  
into pairs of similar triangles.

CONSTRUCTION. Take any point  $L$  within the polygon  $ABCDE$ , and draw  $LA$ ,  $LB$ ,  $LC$ ,  $LD$ ,  $LE$ .

Within the polygon  $PQRST$ , draw  $PX$ ,  $QX$  making the angles  $QPX$ ,  $PQX$  equal to the angles  $BAL$ ,  $ABL$  respectively, and draw  $XR$ ,  $XS$ ,  $XT$ .



PROOF. Because the triangles  $LAB$ ,  $XPQ$  are equiangular to one another, (Constr.)

$LB$  is to  $XQ$  as  $AB$  to  $PQ$ ; (Prop. 4.)

and because the polygons  $ABCDE$ ,  $PQRST$  are similar,

$AB$  is to  $PQ$  as  $BC$  to  $QR$ ; (Def. 2.)

therefore  $LB$  is to  $XQ$  as  $BC$  to  $QR$ . (V. Prop. 5.)

Again because the polygons  $ABCDE$ ,  $PQRST$  are similar,

the angle  $ABC$  is equal to the angle  $PQR$ ;

and the angle  $ABL$  is equal to the angle  $PQX$ ; (Constr.)

therefore the angle  $LBC$  is equal to the angle  $XQR$ .

Therefore the triangles  $LBC$ ,  $XQR$  are equiangular to one another, (Prop. 6.)

and therefore similar. (Prop. 4.)

Similarly it can be proved that the triangles  $LCD$ ,  $LDE$ ,  $LEA$  are similar to the triangles  $XRS$ ,  $XST$ ,  $XTP$  respectively.

Again, because  $AB$  is to  $PQ$  as  $BC$  to  $QR$ ;  
 and because the triangle  $LAB$  is to the triangle  $XPQ$  in  
 the ratio duplicate of the ratio of  $AB$  to  $PQ$ ,  
 and the triangle  $LBC$  is to the triangle  $XQR$  in the ratio  
 duplicate of the ratio of  $BC$  to  $QR$ , (Prop. 19.)  
 therefore the triangle  $LAB$  is to the triangle  $XPQ$  as the  
 triangle  $LBC$  to the triangle  $XQR$ . (V. Prop. 14. Coroll.)

Similarly it can be proved that each of the ratios of the  
 triangles  $LCD$ ,  $LDE$ ,  $LEA$  to the triangles  $XRS$ ,  $XST$ ,  
 $XTP$  respectively is equal to the ratio of the triangle  $LAB$   
 to the triangle  $XPQ$ .

Therefore the polygon  $ABCDE$  is to the polygon  $PQRST$   
 as the triangle  $LAB$  to the triangle  $XPQ$ . (V. Prop. 6.)

Wherefore, a pair of similar polygons &c.

COROLLARY. *Similar polygons are to one another in the  
 ratio duplicate of the ratio of two corresponding sides.*

### EXERCISES.

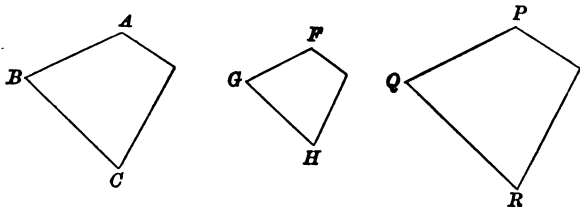
1. If  $ABC$  be a right-angled triangle and  $CD$  be drawn perpendicular to the hypotenuse, then  $AD$  is to  $DB$  as the square on  $AC$  to the square on  $CB$ .

2. If a straight line be drawn from each corner of a square to the nearer point of trisection of the next side of the square in order, so as to form a square, this square will be two-fifths of the original square. What will be the area of the new square, if the lines be drawn to the further point of trisection?

## PROPOSITION 21.

*Polygons which are similar to the same polygon are similar to one another.*

Let each of the polygons  $ABC\dots$ ,  $FGH\dots$ , be similar to the polygon  $PQR\dots$ :  
it is required to prove that  $ABC\dots$ ,  $FGH\dots$  are similar to one another.



**PROOF.** Because the polygons  $ABC\dots$ ,  $PQR\dots$  are similar,

the angle  $ABC$  is equal to the angle  $PQR$ ,  
and  $AB$  is to  $PQ$  as  $BC$  to  $QR$ ; (Def. 2.)

and because the polygons  $FGH\dots$ ,  $PQR\dots$  are similar,  
the angle  $FGH$  is equal to the angle  $PQR$ ,  
and  $PQ$  is to  $FG$  as  $QR$  to  $GH$ .

Therefore the angle  $ABC$  is equal to the angle  $FGH$ ,  
and  $AB$  is to  $FG$  as  $BC$  to  $GH$ . (V. Prop. 14.)

Similarly it can be proved that every pair of corresponding angles of the polygons  $ABC\dots$ ,  $FGH\dots$  are equal and that the ratios of all pairs of corresponding sides are equal.

Therefore the polygons  $ABC\dots$ ,  $FGH\dots$  are similar.

Wherefore, *polygons which are similar &c.*

## EXERCISES.

1. Prove that, if  $ABCD$ ,  $EFGH$  be two quadrilaterals which are equiangular to one another and are such that the ratios  $AB$  to  $EF$ , and  $BC$  to  $FG$  are equal, the quadrilaterals are similar.

2. Prove that, if  $ABCD$ ,  $EFGH$  be two quadrilaterals, which are equiangular to one another and are such that the ratios of  $AB$  to  $EF$  and  $CD$  to  $GH$  are equal, the quadrilaterals are similar.

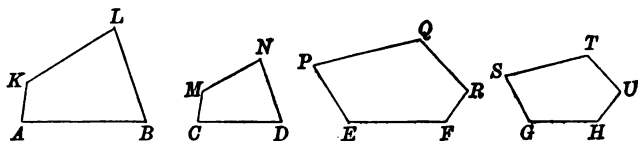
What exceptional case may occur?

3. Prove that, if  $ABCD$ ,  $EFGH$  be two quadrilaterals such that the angles  $ABC$ ,  $BCD$  are equal to the angles  $EFG$ ,  $FGH$  respectively, and the ratios of  $AB$  to  $EF$ ,  $BC$  to  $FG$  and  $CD$  to  $GH$  are all equal, the quadrilaterals are similar.

## PROPOSITION 22. PART 1.

*If four straight lines be proportionals, the ratio of two similar polygons similarly described on the first pair is equal to the ratio of two similar polygons similarly described on the second pair.*

Let the four straight lines  $AB$ ,  $CD$ ,  $EF$ ,  $GH$  be proportionals, and let  $AKLB$ ,  $CMND$  be two similar polygons similarly described on  $AB$ ,  $CD$ , and  $EPQRF$ ,  $GSTUH$  be two similar polygons similarly described on  $EF$ ,  $GH$ : it is required to prove that  $AKLB$  is to  $CMND$  as  $EPQRF$  to  $GSTUH$ .



**PROOF.** Because  $AB$  is to  $CD$  as  $EF$  to  $GH$ ,  
 and  $AKLB$  has to  $CMND$  the ratio duplicate of the ratio  
 of  $AB$  to  $CD$ , (Prop. 20, Coroll.)  
 and  $EPQRF$  has to  $GSTUH$  the ratio duplicate of the ratio  
 of  $EF$  to  $GH$ ,  
 therefore  $AKLB$  is to  $CMND$  as  $EPQRF$  to  $GSTUH$ .  
 (V. Prop. 14, Coroll.)

Wherefore, if four straight lines &c.

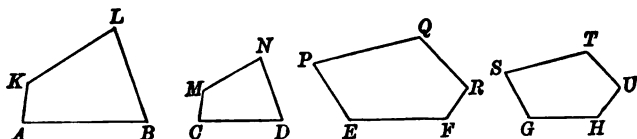
## EXERCISE.

1. Perpendiculars are let fall from two opposite angles of a rectangle on a diagonal: shew that they will divide the diagonal into equal parts, if the square on one side of the rectangle be double that on the other.

## PROPOSITION 22. PART 2.

*If the ratio of two similar polygons similarly described on the first and the second of four straight lines be equal to the ratio of two similar polygons similarly described on the third and the fourth, the four straight lines are proportionals.*

Let  $AB, CD, EF, GH$  be four given straight lines, and let  $AKLB, CMND$  be two similar polygons similarly described on  $AB, CD$ , and  $EPQRF, GSTUH$  be two similar polygons similarly described on  $EF, GH$ , and let  $AKLB, CMND, EPQRF, GSTUH$  be proportionals:  
it is required to prove that  $AB$  is to  $CD$  as  $EF$  to  $GH$ .



PROOF. Because  $AKLB$  has to  $CMND$  the ratio duplicate of the ratio of  $AB$  to  $CD$ , (Prop. 20, Coroll.) and  $EPQRF$  has to  $GSTUH$  the ratio duplicate of the ratio of  $EF$  to  $GH$ ,

and  $AKLB$  is to  $CMND$  as  $EPQRF$  to  $GSTUH$ ,  
therefore  $AB$  is to  $CD$  as  $EF$  to  $GH$ . (V. Prop. 16.)

Wherefore, *if the ratio &c.*

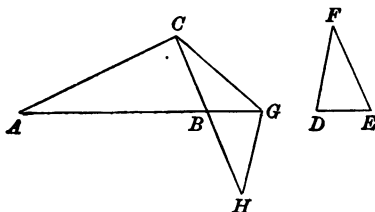
## PROPOSITION 23.

*If two triangles have an angle of the one equal to an angle of the other, the ratio of the areas of the triangles is equal to the ratio compounded of the ratios of the sides about the equal angles.*

Let the triangles  $ABC$ ,  $DEF$  have the angles at  $B$  and  $E$  equal:

it is required to prove that the ratio of the triangle  $ABC$  to the triangle  $DEF$  is equal to the ratio compounded of the ratios  $AB$  to  $DE$  and  $BC$  to  $EF$ .

CONSTRUCTION. In  $AB$ ,  $CB$  produced cut off  $BG$ ,  $BH$  equal to  $ED$ ,  $EF$ , and draw  $CG$ ,  $GH$ .



PROOF. Because in the triangles  $GBH$ ,  $DEF$ ,  
 $BG$  is equal to  $ED$ , and  $BH$  to  $EF$ ,  
 and the angle  $GBH$  to the angle  $DEF$ ,  
 the triangles are equal in all respects. (I. Prop. 4.)  
 And because the triangle  $ABC$  is to the triangle  $BGC$  as  
 $AB$  to  $BG$ , (Prop. 1.)  
 and the triangle  $GCB$  is to the triangle  $GBH$  as  $CB$  to  $BH$ ;  
 therefore the triangle  $ABC$  has to the triangle  $GBH$  the  
 ratio compounded of the ratios  $AB$  to  $BG$  and  $CB$  to  $BH$ ;  
 (V. Def. 8.)  
 therefore the triangle  $ABC$  has to the triangle  $DEF$  the  
 ratio compounded of the ratios  $AB$  to  $DE$  and  $BC$  to  $EF$ .  
 Wherefore, if two triangles &c.

COROLLARY. *If two parallelograms have an angle of the one equal to an angle of the other, the ratio of the areas of the parallelograms is equal to the ratio compounded of the ratios of the sides about the equal angles.*



It is proved in Proposition 23 that the ratio of the triangle  $ABC$  to the triangle  $DEF$  is equal to the ratio compounded of the ratios  $AB$  to  $DE$  and  $BC$  to  $EF$ . Similarly it can be proved that the ratio of the triangle  $ABC$  to the triangle  $DEF$  is equal to the ratio compounded of the ratios  $BC$  to  $EF$  and  $AB$  to  $DE$ . And since any two ratios can be represented by the ratios  $AB$  to  $DE$  and  $BC$  to  $EF$ , if the lines be chosen of proper lengths, it follows that *the magnitude of the ratio compounded of two given ratios is independent of the order in which they are compounded.*

Again, because the proof of Proposition 23 is applicable to two right-angled triangles, we may assume the equal angles at  $B$  and  $E$  to be right angles, in which case the triangle  $ABC$  is equal to half the rectangle  $AB, BC$  and the triangle  $DEF$  is equal to half the rectangle  $DE, EF$ . It follows that the ratio compounded of  $AB$  to  $DE$ , and  $BC$  to  $EF$  is equal to the ratio of the rectangle  $AB, BC$  to the rectangle  $DE, EF$ , or, in other words, *the ratio compounded of the ratios of two pairs of lines is equal to the ratio of the rectangle contained by the antecedents to the rectangle contained by the consequents.*

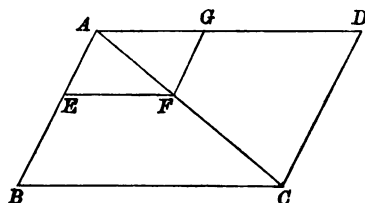
### EXERCISES.

1.  $A$  and  $B$  are two given points;  $AC$  and  $BD$  are perpendicular to a given straight line  $CD$ ;  $AD$  and  $BC$  intersect at  $E$ , and  $EF$  is perpendicular to  $CD$ ; shew that  $AF$  and  $BF$  make equal angles with  $CD$ .
2. If a triangle inscribed in another have one side parallel to a side of the other, its area is to that of the larger triangle as the rectangle contained by the segments of either of the other sides of the original triangle is to the square on that side.
3. If on two straight lines  $OABC, OFED$ , the points be so chosen that  $AE$  is parallel to  $BD$ , and  $AF$  parallel to  $CD$ , then also  $BF$  is parallel to  $CE$ .
4. Find the greatest triangle which can be inscribed in a given triangle so as to have one side parallel to one of the sides of the given triangle.
5. Find the least triangle which can be described about a given triangle.

## PROPOSITION 24.

*A parallelogram about a diagonal of another parallelogram is similar to it.*

Let the parallelogram  $AEFG$  be about the diagonal  $AC$  of the parallelogram  $ABCD$ :  
it is required to prove that  $AEFG$  is similar to  $ABCD$ .



**PROOF.** Because  $EF$  is parallel to  $BC$ ,  
the angles  $AEF$ ,  $AFE$  are equal to the angles  $ABC$ ,  $ACB$  respectively, (I. Prop. 29.)  
and therefore the triangles  $AEF$ ,  $ABC$  are equiangular to one another;  
therefore the parallelograms  $AEFG$ ,  $ABCD$  are equiangular to one another.

And because the triangles  $AEF$ ,  $ABC$  are equiangular to one another,

$AE$  is to  $AB$  as  $EF$  to  $BC$ ; (Prop. 4.)  
and  $EF$  is equal to  $AG$ , and  $BC$  to  $AD$ ; (I. Prop. 34.)  
therefore also  $AE$  is to  $AB$  as  $AG$  to  $AD$ .

Similarly it can be proved that the ratios of all pairs of corresponding sides of the parallelograms  $AEFG$ ,  $ABCD$  are equal.

Therefore the parallelograms are similar.

Wherefore, a parallelogram &c.

## EXERCISES.

1. Prove that, in the figure of VI. 24,  $EG$  and  $BD$  are parallel.
2. Prove that, if in the figure of VI. 24,  $EF$ ,  $GF$  produced cut  $CD$ ,  $CB$  in  $H$ ,  $K$ , then  $HG$ ,  $CA$ ,  $KE$  meet in a point.
3. Prove that, if two similar quadrilaterals  $ABCD$ ,  $A'E'F'G'$  be so placed that  $ABE$ ,  $ADG$  are straight lines, then the points  $A$ ,  $F$ ,  $C$  lie on a straight line.
4. In a given triangle inscribe a rhombus which shall have one of its angular points at a given point in the base, and a side on that base.
5. Construct a parallelogram similar to a given parallelogram, so that two of its vertices are on one side of a given triangle and the other vertices on the other two sides.

## PROPOSITION 25.

*To construct a polygon similar to a given polygon and equal to another given polygon.*

Let  $ABCDE$  be one given polygon, and  $FGHK$  another: it is required to construct a polygon similar to  $ABCDE$  and equal to  $FGHK$ .

CONSTRUCTION. Construct on  $AB$  a rectangle  $AL$  equal to  $ABCDE$ ,

and on  $BL$  construct a rectangle  $LM$  equal to  $FGHK$ .

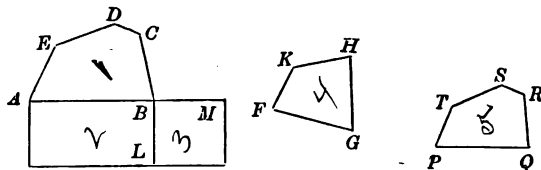
(I. Prop. 45.)

Find  $PQ$  a mean proportional between  $AB$  and  $BM$ ,

(Prop. 13.)

and on  $PQ$  construct a polygon  $PQRST$  similar to  $ABCDE$ , so that  $PQ, AB$  are corresponding sides: (Prop. 18.)

then  $PQRST$  is a polygon constructed as required.



PROOF. Because  $ABCDE$  is to  $PQRST$  in the ratio duplicate of the ratio of  $AB$  to  $PQ$ , and  $AB$  is to  $BM$  in the ratio duplicate of the ratio of  $AB$  to  $PQ$ ;

therefore  $ABCDE$  is to  $PQRST$  as  $AB$  to  $BM$ ; and  $AB$  is to  $BM$  as the rectangle  $AL$  to the rectangle  $LM$ , that is, as  $ABCDE$  to  $FGHK$ .

Therefore  $PQRST$  is equal to  $FGHK$ ; and it was constructed similar to  $ABCDE$ .

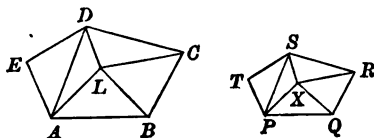
Wherefore, a polygon  $PQRST$  has been constructed similar to the polygon  $ABCDE$  and equal to the polygon  $FGHK$ .

## EXERCISES.

1. Construct a square equal to a given equilateral triangle.
2. Construct an equilateral triangle equal to a given rectangle.

In Proposition 4 it was proved that, if two triangles be equiangular to one another, they are similar. Hence the condition of the equality of the ratios of corresponding sides, which appears in Definition 2, is unnecessary in the case of two triangles, which are equiangular to one another.

If we take the case of two polygons  $ABCDE$ ,  $PQRST$  of more than three sides, which are equiangular to one another, and which are such that all but two of the ratios of pairs of corresponding sides are equal, say  $AB$  to  $PQ$ ,  $BC$  to  $QR$ ,  $CD$  to  $RS$ , where two adjacent sides are omitted, we can prove that the polygons are similar.



Take any point  $L$  within  $ABCD$ , and draw  $LA$ ,  $LB$ ,  $LC$ ,  $LD$ ,  $LE$ . Within the polygon  $PQRS$  draw  $PX$ ,  $QX$ , making the angles  $QPX$ ,  $PQX$  equal to the angles  $BAL$ ,  $ABL$  respectively, and draw  $XR$ ,  $XS$ ,  $SP$ .

It can be proved, as in Proposition 20, that the triangles  $ALB$ ,  $BLC$ ,  $CLD$  are similar to the triangles  $PXQ$ ,  $QXR$ ,  $RXS$  respectively; therefore the angles  $ALB$ ,  $BLC$ ,  $CLD$  are equal to the angles

$PXQ$ ,  $QXR$ ,  $RXS$  respectively, and

therefore the angle  $ALD$  is equal to the angle  $PXS$ ;

also each of the ratios  $LA$  to  $XP$ ,  $LB$  to  $XQ$ ,  $LC$  to  $XR$  and  $LD$  to  $XS$  is equal to the ratio of  $AB$  to  $PQ$ , and therefore  $AL$  is to  $PX$  as  $LD$  to  $XS$ .

Therefore the triangles  $ALD$ ,  $PXS$  are similar, (Prop. 6.) and  $AD$  is to  $PS$  as  $LA$  to  $XP$ .

Hence the two triangles  $AED$ ,  $PTS$  are equiangular to one another; therefore they are similar, and each of the ratios  $DE$  to  $ST$ ,  $EA$  to  $TP$  is equal to the ratio of  $AD$  to  $PS$ , (Prop. 4) which is equal to the ratio of  $LA$  to  $XP$ , and therefore to the ratio  $AB$  to  $PQ$ .

In this case therefore the two polygons are similar.

This method reduces the case, where the two sides whose ratios are omitted are adjacent, to the similar case of quadrilaterals (Ex. 1, page 395). A similar method will reduce the case, where the two sides whose ratios are omitted are not adjacent, to the similar case of quadrilaterals (Ex. 2, page 395).

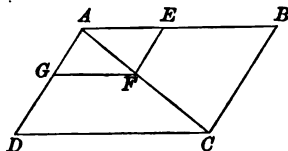
The two cases together justify the remark on Definition 2, page 350.

## PROPOSITION 26.

*If two similar parallelograms have a common angle and be similarly placed, one is about the diagonal of the other.*

Let the parallelograms  $ABCD$ ,  $AEFG$  be similar and similarly placed and have a common angle at  $A$ : it is required to prove that the points  $A$ ,  $F$ ,  $C$  lie on a straight line.

CONSTRUCTION. Draw  $AF$  and  $AC$ .



PROOF. Because the parallelograms  $AEFG$ ,  $ABCD$  are similar,

$AG$  is to  $AD$  as  $GF$  to  $DC$ ; (Def. 2.)

and the angle  $AGF$  is equal to the angle  $ADC$ ;

therefore the triangles  $AGF$ ,  $ADC$  are equiangular to one another. (Prop. 6.)

Therefore the angle  $GAF$  is equal to the angle  $DAC$ ,  
i.e. the three points  $A$ ,  $F$ ,  $C$  lie on a straight line.

Wherefore, if two similar parallelograms &c.

## EXERCISE.

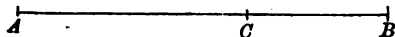
1. Inscribe in a given triangle a parallelogram similar to a given parallelogram so as to have two corners on one side and one on each of the other sides of the triangle.

## PROPOSITION 30.

*To divide a given straight line in extreme and mean ratio.*

Let  $AB$  be the given straight line:  
it is required to divide it in extreme and mean ratio.

CONSTRUCTION. Divide  $AB$  at the point  $C$  into two parts so that the rectangle  $AB, BC$  may be equal to the square on  $AC$ .  
(II. Prop. 11.)



PROOF. Because the rectangle  $AB, BC$  is equal to the square on  $AC$ ,

$AB$  is to  $AC$  as  $AC$  to  $BC$ . (Prop. 17.)

Wherefore, *the given straight line  $AB$  has been divided at  $C$  in extreme and mean ratio.*

## EXERCISES.

1. Two diagonals of a regular pentagon which meet within the figure divide each other in extreme and mean ratio.

2. Divide a given straight line into two parts so that any triangle described on the first part may have to a similar and similarly described triangle on the second part the ratio which the whole has to the second part.

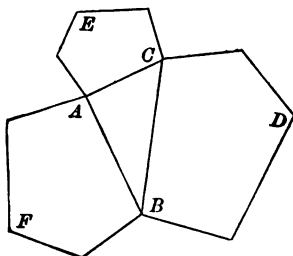
## PROPOSITION 31.

*A polygon on the hypotenuse of a right-angled triangle is equal to the sum of the polygons similarly described on the other sides.*

Let  $ABC$  be a right-angled triangle having the right angle  $BAC$ :

and let  $BDC$ ,  $CEA$ ,  $AFB$  be similar polygons similarly described on  $BC$ ,  $CA$ ,  $AB$  respectively:

it is required to prove that the polygon  $BDC$  is equal to the sum of the polygons  $CEA$ ,  $AFB$ .



**PROOF.** Because  $BDC$  has to  $CEA$  the ratio duplicate of the ratio of  $BC$  to  $CA$ ,  
 and the square on  $BC$  has to the square on  $CA$  the ratio duplicate of the ratio of  $BC$  to  $CA$ ; (Prop. 20, Coroll.)  
 therefore  $BDC$  is to  $CEA$  as the square on  $BC$  to the square on  $CA$ ;  
 therefore  $BDC$  is to the square on  $BC$  as  $CEA$  to the square on  $CA$ .  
(V. Prop. 9.)



Similarly it can be proved that  
 $BDC$  is to the square on  $BC$  as  $AFB$  to the square on  $AB$ .  
 Therefore  $BDC$  is to the square on  $BC$   
 as the sum of  $CEA$ ,  $AFB$  to the sum of the squares on  $CA$ ,  
 $AB$ ; (V. Prop. 6.)  
 and the square on  $BC$  is equal to the sum of the squares  
 on  $CA$ ,  $AB$ ; (I. Prop. 47.)  
 therefore  $BDC$  is equal to the sum of  $CEA$ ,  $AFB$ .  
 Wherefore, a polygon &c.

## EXERCISES.

1. Divide a given finite straight line into two parts so that the squares on them shall be to one another in a given ratio.
2. Construct an equilateral triangle equal to the sum of two given equilateral triangles.
3. On two given lines similar triangles are described; construct a similar triangle equal to the difference of the given triangles.
4. Construct a triangle equal to the sum of three given similar triangles and similar to them.
5. Construct a polygon equal to the sum of any number of similar polygons and similar to them.

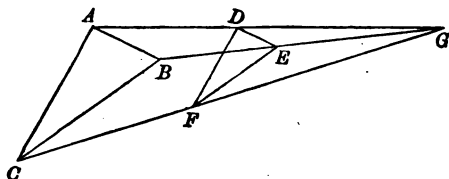
## PROPOSITION 32.

*If two triangles have sides parallel in pairs, the straight lines joining the corresponding vertices meet in a point.*

Let  $ABC$ ,  $DEF$  be two triangles such that the sides  $BC$ ,  $CA$ ,  $AB$  are parallel to the sides  $EF$ ,  $FD$ ,  $DE$  respectively:

it is required to prove that the straight lines joining the pairs of points  $A$ ,  $D$ ;  $B$ ,  $E$ ;  $C$ ,  $F$  meet in a point.

CONSTRUCTION. Draw  $AD$ ,  $BE$  and let them meet in  $G$ ; and draw  $GC$ ,  $GF$ .



PROOF. Because  $AB$  is parallel to  $DE$ ,  
the angles  $GAB$ ,  $GBA$  are equal to the angles  $GDE$ ,  $GED$   
respectively; (I. Prop. 29.)

therefore the triangles  $GAB$ ,  $GDE$  are equiangular to one another; (I. Prop. 32.)

therefore  $GB$  is to  $GE$  as  $BA$  to  $ED$ ; (Prop. 4.)

and because the triangles  $ABC$ ,  $DEF$  are equiangular to one another, (I. Prop. 34, Coroll. 2.)

$BA$  is to  $ED$  as  $BC$  to  $EF$ ; (Prop. 4.)

therefore  $GB$  is to  $GE$  as  $BC$  to  $EF$ ; (V. Prop. 5.)

and the angle  $GBC$  is equal to the angle  $GEF$ ; (I. Prop. 29.)

therefore the triangles  $GBC$ ,  $GEF$  are equiangular to one another; (Prop. 6.)

therefore the angles  $BGC$ ,  $EGF$  are equal,

that is, the points  $C$ ,  $F$ ,  $G$  lie on a straight line,

or, in other words,  $AD$ ,  $BE$ ,  $CF$  meet in a point.

Wherefore, if two triangles &c.

It will be seen at once that, if in the diagram of Proposition 32  $AB$  be equal to  $DE$ , then the straight lines  $AD$ ,  $BE$  do not meet at any point at a finite distance, in other words, they are parallel. Also, because the triangles  $ABC$ ,  $DEF$  are similar, if  $AB$  be equal to  $DE$ , then also  $BC$  is equal to  $EF$ , and therefore  $BE$  and  $CF$  are parallel.

Hence we must consider the case when the two triangles are similarly placed and equal as a special case in which the point of intersection of the lines joining the corresponding vertices is at an infinite distance.

## EXERCISES.

1. If two similar triangles be similarly placed on two parallel straight lines, the lines joining corresponding vertices meet in a point.

2. If any two similar polygons have three pairs of corresponding sides parallel, the straight lines joining the corresponding vertices meet in a point.

3.  $AB$  is a fixed diameter of a circle  $ABC$ :  $PQ$  is a straight line parallel to  $AB$  and of constant length, which moves so that its middle point traces out the circle  $ABC$ ; find the locus of the intersection of  $AP$ ,  $BQ$  and of  $AQ$ ,  $BP$ .

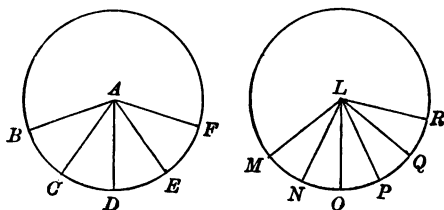
4. Prove that, if the corresponding sides of  $ABCD$ ,  $EFGH$  two squares be parallel, the straight lines  $AE$ ,  $BF$ ,  $CG$ ,  $DH$  pass through a point, and  $AG$ ,  $BH$ ,  $CE$ ,  $DF$  pass through another point.

## PROPOSITION 33. PART 1.

*In equal circles angles at the centres have the same ratio as the arcs on which they stand.*

Let  $BCD$ ,  $MNO$  be two given equal circles, and let  $BAC$ ,  $MLN$  be two angles at their centres:  
it is required to prove that the angle  $BAC$  is to the angle  $MLN$  as the arc  $BC$  to the arc  $MN$ .

CONSTRUCTION. From  $A$  draw any number of radii  $AD$ ,  $AE$ ,  $AF$  making the angles  $CAD$ ,  $DAE$ ,  $EAF$  each equal to the angle  $BAC$ ; and from  $L$  draw any number of radii  $LO$ ,  $LP$ ,  $LQ$ ,  $LR$  making the angles  $NLO$ ,  $OLP$ ,  $PLQ$ ,  $QLR$  each equal to the angle  $MLN$ .



PROOF. Because the angles  $BAC$ ,  $CAD$ ,  $DAE$ ,  $EAF$  are all equal,  
the arcs  $BC$ ,  $CD$ ,  $DE$ ,  $EF$  are all equal; (III. Prop. 26.)  
therefore the angle  $BAF$  and the arc  $BF$  are equimultiples  
of the angle  $BAC$  and the arc  $BC$ .

Similarly it can be proved that  
the angle  $MLR$  and the arc  $MR$  are equimultiples of the  
angle  $MLN$  and the arc  $MN$ .

And, because the circles are equal, if the angle  $BAF$  be  
equal to the angle  $MLR$ ,

the arc  $BF$  is equal to the arc  $MR$ , (III. Prop. 26.)  
and if the angle  $BAF$  be greater or less than the angle  $MLR$ ,  
the arc  $BF$  is greater or less respectively than the arc  $MR$ .  
Therefore the angle  $BAC$  is to the angle  $MLN$  as the arc  
 $BC$  to the arc  $MN$ . (V. Def. 5.)

Wherefore, in equal circles &c.

COROLLARY. *In equal circles angles at the circumferences have the same ratio as the arcs on which they stand.*

The angles at the centres are double of the angles at the circumferences, and therefore have the same ratio.

(V. Prop. 6, Coroll.)

In the construction of Proposition 33 there is nothing to limit the magnitude of the multiple angles  $BAF$ ,  $MLR$ ; they may be greater than two right angles, greater than four right angles, or greater than any multiple of four right angles, and at the same time the multiple arcs  $BF$ ,  $MR$  will be greater than half the circumference of the circle, greater than the circumference, or greater than any multiple of the circumference.

In the Third Book (page 221) we had occasion to remark that the admittance of angles equal to or greater than two right angles was not inconsistent with Euclid's methods. We may now go further and say that the admittance of angles without any restriction whatever on their magnitude is essential to his method. The validity of the proof of this Proposition depends on the possibility of choosing any multiples we please of the angles  $BAC$ ,  $MLN$ , that is, of taking the multiple angles  $BAF$ ,  $MLR$  as large as we please.

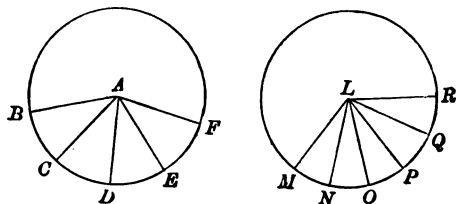
## PROPOSITION 33. PART 2.

*In equal circles, the areas of sectors have the same ratio as their angles.*

Let  $BCD$ ,  $MNO$  be two given equal circles, and let  $BAC$ ,  $MLN$  be two angles at their centres:

it is required to prove that the angle  $BAC$  is to the angle  $MLN$  as the sector  $BAC$  to the sector  $MLN$ .

CONSTRUCTION. From  $A$  draw any number of radii  $AD$ ,  $AE$ ,  $AF$  making the angles  $CAD$ ,  $DAE$ ,  $EAF$  each equal to the angle  $BAC$ ; and from  $L$  draw any number of radii  $LO$ ,  $LP$ ,  $LQ$ ,  $LR$  making the angles  $NLO$ ,  $OLP$ ,  $PLQ$ ,  $QLR$  each equal to the angle  $MLN$ .



PROOF. Because the angle  $CAD$  is equal to the angle  $BAC$ , it is possible to shift the figure  $CAD$  so that  $AC$  will be on  $AB$ , and  $AD$  on  $AC$ ; if this be done, then the point  $C$  will coincide with  $B$  and  $D$  with  $C$ , and therefore the arc  $CD$  with the arc  $BC$ . (III. Prop. 23.) Therefore the sector  $CAD$  coincides with the sector  $BAC$  and is equal to it in all respects.

Similarly it can be proved that the sectors  $DAE$ ,  $EAF$  are each equal to the sector  $BAC$ .

Therefore the angle  $BAF$  and the sector  $BAF$  are equimultiples of the angle  $BAC$  and the sector  $BAC$ .

Similarly it can be proved that the angle  $MLR$  and the sector  $MLR$  are equimultiples of the angle  $MLN$  and the sector  $MLN$ .

And it can be proved as before that, if the angle  $BAF$  be equal to the angle  $MLR$ ,

the sector  $BAF$  is equal to the sector  $MLR$ ;  
and, if the angle  $BAF$  be greater or less than the angle  $MLR$ ,  
the sector is greater or less respectively than the sector  $MLR$ ;  
therefore the angle  $BAC$  is to the angle  $MLN$  as the sector  
 $BAC$  to the sector  $MLN$ . (V. Def. 5.)

Wherefore, *in equal circles &c.*

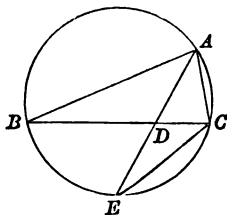
COROLLARY. *In equal circles the areas of sectors have the same ratio as the arcs on which they stand.*

## PROPOSITION 34.

*If an angle of a triangle be bisected by a straight line which cuts the opposite side, the rectangle contained by the segments of that side is less than the rectangle contained by the other sides by the square on the line.*

Let the angle  $BAC$  of the triangle  $ABC$  be bisected by the straight line  $AD$ , which cuts  $BC$  at  $D$ : it is required to prove that the rectangle  $BD, DC$  is less than the rectangle  $BA, AC$  by the square on  $AD$ .

CONSTRUCTION. Describe the circle  $ABC$ ; (IV. Prop. 5.) produce  $AD$  to meet the circle at  $E$ , and draw  $EC$ .



PROOF. Because in the triangles  $BAD, EAC$ ,  
the angle  $BAD$  is equal to the angle  $EAC$ ,  
(Hypothesis.)  
and the angle  $ABD$  to the angle  $AEC$ ,  
(III. Prop. 21.)

therefore the triangles are equiangular to one another ;

therefore  $BA$  is to  $EA$  as  $AD$  to  $AC$  ; (Prop. 4.)

therefore the rectangle  $BA, AC$  is equal to the rectangle  $EA, AD$ ,  
(Prop. 16.)

that is, to the rectangle  $ED, DA$  together with the square on  $AD$ .  
(II. Prop. 3.)

And the rectangle  $ED, DA$  is equal to the rectangle  $BD, DC$  ;  
(III. Prop. 35.)

therefore the rectangle  $BD, DC$  is less than the rectangle  $BA, AC$  by the square on  $AD$ .

Wherefore, if an angle &c.



## EXERCISES.

1. If an angle of a triangle be bisected externally by a straight line which cuts the opposite side produced, the rectangle contained by the segments of that side is greater than the rectangle contained by the other sides by the square on the line.

2. Prove that, if the internal and the external bisectors of the vertical angle of a triangle  $ABC$  cut  $BC$  in  $D$  and  $E$ , then the square on  $DE$  is equal to the difference of the rectangles  $EB, EC$  and  $DB, DC$ .

3. If  $I$  be the centre of the inscribed circle of a triangle  $ABC$  and  $AI$  produced cut the circumscribed circle  $ABC$  in  $E$ , then the rectangle contained by  $AI, IE$  is equal to twice the rectangle contained by the radii of the circumscribed and the inscribed circles.

(See Ex. 46, page 324.)

4. If  $I_1$  be the centre of the circle of the triangle  $ABC$  escribed beyond  $BC$  and  $AI_1$  cut the circumscribed circle  $ABC$  in  $E$ , then the rectangle contained by  $AI_1, I_1E$  is equal to twice the rectangle contained by the radii of the circumscribed and the escribed circles.

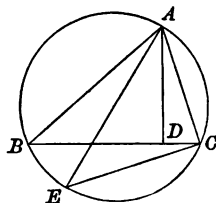
5.  $AB$  is the base of a triangle  $ABC$  whose sides are segments of a line divided in extreme and mean ratio.  $CP$  the bisector of the angle  $C$ , and  $CQ$  the perpendicular from  $C$  on  $AB$  meet  $AB$  in  $P$  and  $Q$ . Prove that the square on  $CP$  is equal to twice the rectangle contained by  $PQ$  and  $AB$ .

## PROPOSITION 35.

*If a perpendicular be drawn from a vertex of a triangle to the opposite side, the rectangle contained by the other sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle described about the triangle.*

Let  $AD$  be the perpendicular drawn from the vertex  $A$  of the given triangle  $ABC$  to the opposite side  $BC$ : it is required to prove that the rectangle contained by  $AB, AC$  is equal to the rectangle contained by  $AD$  and the diameter of the circle described about  $ABC$ .

CONSTRUCTION. Describe the circle  $ABC$ ; (IV. Prop. 5.) draw the diameter  $AE$  and draw  $EC$ .



PROOF. Because in the triangles  $BAD, EAC$ ,  
the angle  $ABD$  is equal to the angle  $AEC$ , (III. Prop. 21.)  
and the angle  $ADB$  to the angle  $ACE$ ; (III. Prop. 31.)  
therefore the triangles are equiangular to one another,  
(I. Prop. 32.)  
and  $BA$  is to  $EA$  as  $AD$  to  $AC$ ; (Prop. 4.)  
therefore the rectangle  $BA, AC$  is equal to the rectangle  
 $EA, AD$ . (Prop. 16.)

Wherefore, if a perpendicular &c.

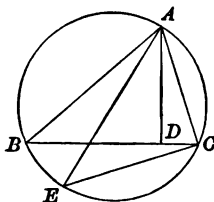
## PROPOSITION 35 A.

*The ratio of twice the area of a triangle to the rectangle contained by two of the sides is equal to the ratio of the third side to the diameter of the circumscribed circle of the triangle.*

Let  $ABC$  be a triangle :

it is required to prove that twice the area of  $ABC$  is to the rectangle contained by  $AC$ ,  $BC$  as  $AB$  to the diameter of the circle described about  $ABC$ .

CONSTRUCTION. Describe the circle  $ABC$  ; draw the diameter  $AE$ , draw  $AD$  perpendicular to  $BC$  and draw  $EC$ .



PROOF. Because in the triangles  $BAD$ ,  $EAC$ ,  
the angle  $ABD$  is equal to the angle  $AEC$ , (III. Prop. 21.)  
and the angle  $ADB$  to the angle  $ACE$  ; (III. Prop. 31.)  
therefore the triangles are equiangular to one another,

(I. Prop. 32.)

and  $AD$  is to  $AC$  as  $AB$  to  $AE$ . (Prop. 4.)

Therefore the rectangle  $AD$ ,  $BC$  is to the rectangle  $AC$ ,  $BC$   
as  $AB$  to  $AE$  ; (Prop. 1.)

and the rectangle  $AD$ ,  $BC$  is equal to twice the area of the  
triangle  $ABC$  ;

therefore twice the area of the triangle  $ABC$  is to the  
rectangle  $AC$ ,  $BC$  as  $AB$  to the diameter of the circle  $ABC$ .

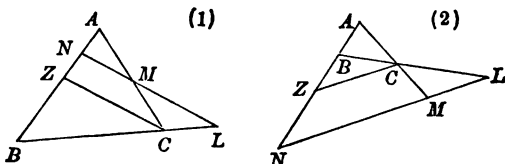
Wherefore, *the ratio &c.*

## ADDITIONAL PROPOSITION 1.

*If a straight line cut the three sides of a triangle produced if necessary, the ratio compounded of the ratios of the segments of the sides taken in order is equal to unity.\**

Let the sides  $BC$ ,  $CA$ ,  $AB$  of the triangle  $ABC$  be cut by the straight line  $LMN$  in  $L$ ,  $M$ ,  $N$  respectively.

Through  $C$  draw  $CZ$  parallel to  $LMN$  to meet  $ABN$  in  $Z$ .



Because  $ZC$ ,  $NML$  are parallel,

$AM$  is to  $MC$  as  $AN$  to  $NZ$ ,

and  $CL$  is to  $LB$  as  $ZN$  to  $NB$ ; (Prop. 2.)

therefore the ratio compounded of

the ratios  $AM$  to  $MC$  and  $CL$  to  $LB$  is equal to the ratio compounded of the ratios  $AN$  to  $NZ$  and  $ZN$  to  $NB$ ,

i.e. the ratio  $AN$  to  $NB$ ; (V. Def. 8.)

therefore the ratio compounded of the ratios  $AM$  to  $MC$ ,  $CL$  to  $LB$  and  $BN$  to  $NA$  is equal the ratio compounded of the ratios  $AN$  to  $NB$  and  $NB$  to  $AN$ , that is, to the ratio  $AN$  to  $AN$ , i.e. to unity.

(V. Def. 2.)

\* This theorem is attributed to Menelaus, a Greek Geometer, who lived in the latter part of the first century A.D.

**DEFINITION.** *A straight line drawn to cut a series of lines is often called a transversal.*

The straight line  $LMN$  in Additional Proposition 1 is a transversal of the triangle  $ABC$ .

### EXERCISES.

1. Points  $E, F$  are taken in the sides  $AC, AB$  of a triangle such that  $AE$  is twice  $EC$  and  $BF$  is twice  $FA$ ;  $FE$  produced cuts  $BC$  in  $D$ ; find the ratio  $BD$  to  $DC$ .

2. If the bisectors of the angles  $B, C$  of a triangle  $ABC$  meet the opposite sides in  $D$  and  $E$ , and if the straight line  $DE$  produced meet  $BC$  produced in  $F$ , then the external angle at  $A$  is bisected by  $AF$ .

3.  $BD$  is the perpendicular let fall from one end of the base upon the straight line bisecting the vertical angle  $BAC$  of a triangle. If  $BA$  be three times as long as  $AC$ ,  $AD$  will be bisected at the point  $E$ , where it cuts the base.

4. If a side  $BC$  of a triangle  $ABC$  be bisected by a straight line which meets the sides  $AB, AC$ , produced if necessary, in  $D$  and  $E$  respectively, then  $AE$  is to  $EC$  as  $AD$  to  $DB$ .

5. If one side of a given triangle be produced and the other shortened by equal quantities, the line joining the points of section will be divided by the base in the inverse ratio of the sides.

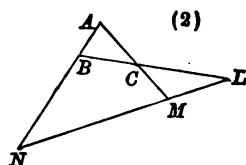
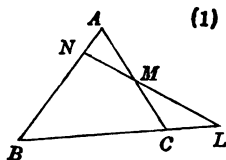
6. In the sides  $AB, AC$  of a triangle  $ABC$  two points  $D, E$  are taken such that  $BD$  is equal to  $CE$ ;  $DE, BC$  are produced to meet at  $F$ : shew that  $AB$  is to  $AC$  as  $EF$  to  $DF$ .

The converse of the theorem on page 418 may be stated as follows:—

ADDITIONAL PROPOSITION 2.

*If three points be taken on the sides of a triangle (either one on a side produced and the other two on sides, or else all three on sides produced), such that the ratio compounded of the ratios of the segments of the sides taken in order is equal to unity, the three points lie on a straight line.*

Let three points  $L, M, N$  be taken on the sides  $BC, CA, AB$  of a triangle  $ABC$ , either all on sides produced (fig. 2) or one  $L$  on a side produced, and the others  $M, N$  on sides (fig. 1) such that the ratio compounded of the ratios  $AM$  to  $MC$ ,  $CL$  to  $LB$  and  $BN$  to  $NA$  is equal to unity.



Draw  $LM$  and let it produced cut  $AB$  in  $P$ .

Then the ratio compounded of the ratios

$AM$  to  $MC$ ,  $CL$  to  $LB$  and  $BP$  to  $PA$  is equal to unity;

(Add. Prop. 1.)

and the ratio compounded of the ratios

$AM$  to  $MC$ ,  $CL$  to  $LB$  and  $BN$  to  $NA$  is equal to unity;

(Hypothesis.)

therefore the ratio  $BP$  to  $PA$  is equal to the ratio  $BN$  to  $NA$ ;

therefore  $BP$  is to  $BA$  as  $BN$  to  $BA$ ; (V. Prop. 10 or 11)

therefore  $BP$  is equal to  $BN$ ; (V. Prop. 3.)

that is,  $P$  coincides with  $N$ ,

or, in other words,  $L, M, N$  are in a straight line.

## EXERCISES.

1. The inscribed circle of a triangle  $ABC$  touches the sides  $BC$ ,  $CA$ ,  $AB$  at  $D$ ,  $E$ ,  $F$ ;  $EF$ ,  $FD$ ,  $DE$  produced meet  $BC$ ,  $CA$ ,  $AB$  in  $L$ ,  $M$ ,  $N$ : prove that  $L$ ,  $M$ ,  $N$  are collinear.

2. An escribed circle of a triangle  $ABC$  touches the side  $BC$  at  $D$  and the sides  $AC$ ,  $AB$  produced at  $E$ ,  $F$ ;  $ED$ ,  $FD$  produced cut  $AB$ ,  $AC$  in  $K$ ,  $H$  respectively; prove that  $FE$ ,  $BC$ ,  $KH$  meet in a point.

3. If  $AB$ ,  $CD$ ,  $EF$  be three parallel straight lines, and  $AC$ ,  $BD$  meet in  $N$ ,  $CE$ ,  $DF$  meet in  $L$ , and  $EA$ ,  $FB$  meet in  $M$ , then  $L$ ,  $M$ ,  $N$  lie on a straight line.

4. If  $D$ ,  $E$ ,  $F$  be the points of contact with  $BC$ ,  $CA$ ,  $AB$  of the inscribed circle, or of any one of the escribed circles of the triangle  $ABC$ , the lines  $AD$ ,  $BE$ ,  $CF$  pass through a point.

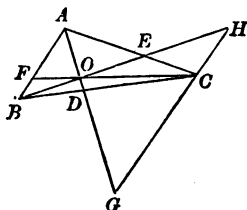
5. If  $D$  be the point of contact of the inscribed circle of a triangle  $ABC$  with  $BC$ , and  $E$ ,  $F$  the points of contact of escribed circles with  $CA$  produced and  $BA$  produced respectively, then  $AD$ ,  $BE$ ,  $CF$  meet in a point.

6. If one escribed circle of a triangle  $ABC$  touch  $AC$  in  $F$  and  $BA$  produced in  $G$  and another escribed circle touch  $AB$  in  $H$  and  $CA$  produced in  $K$ , then  $FH$ ,  $KG$  produced cut  $BC$  produced in points equidistant from the middle point of  $BC$ .

## ADDITIONAL PROPOSITION 3.

*If three straight lines be drawn from the vertices of a triangle meeting in a point and cutting the opposite sides or the sides produced, the ratio compounded of the ratio of the segments of the sides taken in order is equal to unity.\**

Let the straight lines  $AO$ ,  $BO$ ,  $CO$  be drawn from the vertices of the triangle  $ABC$  meeting in  $O$ , and cutting  $BC$ ,  $CA$ ,  $AB$  in  $D$ ,  $E$ ,  $F$  respectively.



Through  $C$  draw  $HCG$  parallel to  $AB$  to meet  $BO$ ,  $AO$  produced in  $H$ ,  $G$ .

Then because the triangles  $AOE$ ,  $GOE$  are equiangular to one another,  
 $AE$  is to  $GE$  as  $EO$  to  $GO$ ; (Prop. 4.)

and because the triangles  $FOB$ ,  $COH$  are equiangular to one another,  
 $FB$  is to  $CH$  as  $FO$  to  $CO$ ;

therefore  $AE$  is to  $GE$  as  $FB$  to  $CH$ , (V. Prop. 5.)

and therefore  $AE$  is to  $FB$  as  $GE$  to  $CH$ . (V. Prop. 9.)

And because the triangles  $CEH$ ,  $AEB$  are equiangular to one another,  
 $CE$  is to  $AE$  as  $CH$  to  $AB$ ;

and because the triangles  $BDA$ ,  $CDG$  are equiangular to one another,  
 $BD$  is to  $CD$  as  $BA$  to  $CG$ ;

therefore the ratio compounded of the ratios

$AE$  to  $FB$ ,  $CE$  to  $EA$ , and  $BD$  to  $DC$

is equal to the ratio compounded of the ratios

$GC$  to  $CH$ ,  $CH$  to  $AB$ , and  $AB$  to  $CG$ ,

which is equal to unity.

\* This theorem was first published in the year 1678 by Giovanni Ceva, an Italian.



In Additional Proposition 3 it has been proved that, if through the vertices  $A, B, C$  of a triangle three concurrent straight lines  $AD, BE, CF$  be drawn meeting the sides  $BC, CA, AB$  in  $D, E, F$ , the ratio of  $BD$  to  $DC$  is equal to the ratio compounded of the ratios  $BF$  to  $FA$  and  $AE$  to  $EC$ .

In Additional Proposition 1 it has been proved that, if the straight line  $FE$  be drawn and produced to meet  $BC$  produced in  $L$ , the ratio  $BL$  to  $LC$  is equal to the ratio compounded of the ratios  $BF$  to  $FA$  and  $AE$  to  $EC$ .

Therefore  $BD$  is to  $DC$  as  $BL$  to  $LC$ ,  
or, in other words,  $BDCL$  is a harmonic range.

It is a remarkable fact that, although the theorem on page 418 had been known as early as the 1st century, the theorem on page 422, which seems a very natural complement to the other, should not have been discovered until the 17th century.

### EXERCISES.

1. Prove Ceva's Theorem for a triangle  $ABC$  and a point  $O$ , (1) when  $O$  lies between  $AB$  produced and  $AC$  produced, (2) when  $O$  lies between  $BA$  produced and  $CA$  produced.

2. Prove Ceva's Theorem by using the result of Ex. 3, page 355.

3. Prove Ceva's Theorem by the use of Menelaus' Theorem, considering in the figure of Add. Prop. 3  $COF$  a transversal of the triangle  $ABD$  and  $BOE$  a transversal of the triangle  $ADC$ .

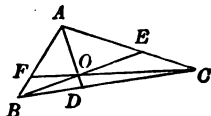
4.  $D, E, F$  are the points in which the bisectors of the angles  $A, B, C$  of a triangle cut the opposite sides; prove that, if  $RC$  be equal to half the sum of the sides  $AB, AC$ , then  $EF$  bisects  $AD$ .

The converse of the theorem on page 422 may be stated as follows;—

#### ADDITIONAL PROPOSITION 4.

*If three straight lines be drawn through the vertices of a triangle cutting the opposite sides (either all three sides, or else one side and the other two sides produced) so that the ratio compounded of the ratios of the segments of the sides taken in order is equal to unity, the three straight lines meet in a point.*

Let three straight lines  $AD$ ,  $BE$ ,  $CF$  be drawn from the vertices  $A$ ,  $B$ ,  $C$  of a triangle  $ABC$  to cut the opposite sides in  $D$ ,  $E$ ,  $F$  respectively, so that the ratio compounded of the ratios  $AF$  to  $FB$ ,  $BD$  to  $DC$  and  $CE$  to  $EA$  is equal to unity.



Let  $AD$ ,  $CF$  meet in  $O$ : draw  $BO$  and produce it to meet  $CA$  in  $P$ .

Then because the ratio compounded of the ratios  $AF$  to  $FB$ ,  $BD$  to  $DC$  and  $CP$  to  $PA$  is equal to unity, (Add. Prop. 3.)

and also the ratio compounded of the ratios  $AF$  to  $FB$ ,  $BD$  to  $DC$  and  $CE$  to  $EA$  is equal to unity; (Hypothesis.)

therefore the ratio  $CP$  to  $PA$  is equal to the ratio  $CE$  to  $EA$ ;

therefore  $CP$  is to  $CA$  as  $CE$  to  $CA$ ; (V. Prop. 10 or 11.)

therefore  $CP$  is equal to  $CE$ ; (V. Prop. 8.)

therefore  $P$  coincides with  $E$ ,

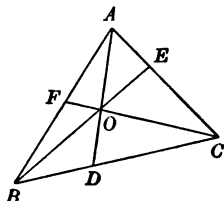
or, in other words,  $AD$ ,  $BE$ ,  $CF$  meet in a point.

If in the sides  $BC$ ,  $CA$ ,  $AB$  of a triangle, points  $D$ ,  $E$ ,  $F$  be taken such that  $BD$  is to  $DC$  as  $n$  to  $m$ ,  $CE$  is to  $EA$  as  $l$  to  $n$ , and  $AF$  is to  $FB$  as  $m$  to  $l$ , where  $l$ ,  $m$ ,  $n$  are any three integers, the straight lines  $AD$ ,  $BE$ ,  $CF$  meet in a point, say  $O$ . (Add. Prop. 4.)

It is proved by Add. Prop. 1 that the ratio of  $AO$  to  $OD$  is equal to the ratio compounded of the ratios  $AE$  to  $EC$  and  $CB$  to  $BD$ , that is, of the ratios  $n$  to  $l$  and  $m+n$  to  $n$ ;

therefore  $AO$  is to  $OD$  as  $m+n$  to  $l$ .

Similarly it appears that  $BO$  is to  $OE$  as  $n+l$  to  $m$  and that  $CO$  is to  $OF$  as  $l+m$  to  $n$ .



It follows that, if we divide  $BC$  in  $D$  so that  $BD$  is to  $DC$  as  $n$  to  $m$ , and then divide  $DA$  in  $O$  so that  $DO$  is to  $OA$  as  $l$  to  $m+n$ , we arrive at the same point, as if we divide  $CA$  in  $E$  so that  $CE$  is to  $EA$  as  $l$  to  $n$ , and then divide  $EB$  in  $O$  so that  $EO$  is to  $OB$  as  $m$  to  $n+l$ , or as if we divide  $AB$  in  $F$  so that  $AF$  is to  $FB$  as  $m$  to  $l$ , and then divide  $FC$  in  $O$  so that  $FO$  is to  $OC$  as  $n$  to  $l+m$ .

This point  $O$  is called the **centroid of weights**  $l$ ,  $m$ ,  $n$  at  $A$ ,  $B$ ,  $C$  respectively. It appears that the position of the centroid of three weights is independent of the order in which the weights are taken, or, in other words, the centroid of three weights is a unique point.

It is not difficult to see that this proposition can be extended to any number of weights, so that we may state the proposition in the general form, *the centroid of a number of given weights is a unique point.*

#### EXERCISE.

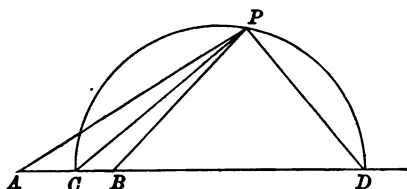
1. From the vertex  $A$  of a triangle  $ABC$  a straight line is drawn cutting  $BC$  in  $D$ , and the angles  $BDA$ ,  $CDA$  are bisected by straight lines cutting  $AB$ ,  $AC$  in  $F$ ,  $E$  respectively: prove that  $AD$ ,  $BE$ ,  $CF$  intersect in a point.

## ADDITIONAL PROPOSITION 5.

*The locus of a point, the ratio of whose distances from two given points is constant, is a circle\*.*

Let  $A, B$  be two given points and  $P$  a point such that the ratio of  $AP$  to  $BP$  is equal to the given ratio.

Draw  $PA, PB$ ; and draw  $PC, PD$  the internal and the external bisectors of the angle  $APB$  meeting  $AB$  in  $C$  and  $AB$  produced in  $D$ .



Because  $PC, PD$  are the bisectors of the angle  $APB$ , therefore the ratios of  $AC$  to  $CB$  and  $AD$  to  $DB$  are equal to the ratio of  $AP$  to  $PB$ ; and the ratio of  $AP$  to  $PB$  is constant;

therefore  $C$  and  $D$  are two fixed points. (Ex. 1, page 359.)

And because  $PC, PD$  are the bisectors of the angle  $APB$ , the angle  $CPD$  is a right angle. (Ex. 5, page 43.)

Therefore every point on the locus of  $P$  must lie on the circle upon the fixed line  $CD$  as diameter.

Next we will prove that every point of the circle belongs to the locus.

Let  $P$  be any point on the circle described on  $CD$  as diameter.

Draw  $PA, PC$ ; and draw  $PE$  making the angle  $CPE$  equal to the angle  $CPA$  and meeting  $CD$  at  $E$ ;

then  $AP$  is to  $PE$  as  $AC$  to  $CE$ . (Prop. 3, Part 1.)

Again, because  $CPD$  is a right angle,

$PD$  is the external bisector of the angle  $APE$ ;

therefore  $AP$  is to  $PE$  as  $AD$  to  $DE$ .

\* This theorem is attributed to Apollonius of Perga, a Greek geometer, who lived in the latter part of the third century B.C.

Therefore  $AC$  is to  $CE$  as  $AD$  to  $DE$ ; (V. Prop. 5.)  
 and  $AC$  is to  $CB$  as  $AD$  to  $DB$ ;  
 therefore  $CE$  is to  $ED$  as  $CB$  to  $BD$ ,  
 and  $E$  coincides with  $B$ , which is a fixed point. (Ex. 1, page 359.)  
 Therefore  $AP$  is to  $PB$  in the fixed ratio  $AC$  to  $CB$  for every point  $P$  on the circle.

We may state the result of this proposition thus:—*If a circle be described upon the straight line joining two conjugate points of a harmonic range as a diameter, the ratio of the distances of a point on the circle from the other pair of conjugate points is constant.*

## EXERCISES.

1. Prove that, if  $A, B$  be two fixed points and  $P$  be a point such that  $PA$  is equal to  $m$  times  $PB$ ,

- (1) if  $m$  vanish, the locus reduces to the point  $A$ ;
- (2) if  $m$  be equal to unity, the locus is the straight line which bisects  $AB$  at right angles;
- (3) if  $m$  be infinitely great, the locus reduces to the point  $B$ ;
- (4) if  $m$  be greater than unity, the locus is a circle excluding  $A$  and including  $B$ ;
- (5) if  $m$  be less than unity, the locus is a circle including  $A$  and excluding  $B$ ;
- (6) if  $m$  be greater than unity, the greater the value of  $m$ , the less the circle;
- (7) if  $m$  be less than unity, the less the value of  $m$ , the less the circle;
- (8) the loci for two different values of  $m$  do not intersect.

2.  $A$  and  $B$  are the centres of two circles. A straight line  $PQ$  parallel to  $AB$  meets the circles in  $P$  and  $Q$ : find the locus of the point of intersection of  $AP$  and  $BQ$ .

3. Find a point such that its distances from three given points may be in given ratios.

4. Prove that, if a map be laid flat on another map of the same district on a larger scale, there is one place in the district which is represented in the two maps by points which are superposed one on the other.

## ADDITIONAL PROPOSITION 6.

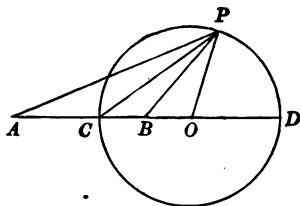
*If the line between one pair of conjugates of a harmonic range be bisected, the square on half the line is equal to the rectangle contained by the segments of the line between the other pair of conjugates made by the point of bisection.*

Let  $ACBD$  be a harmonic range, such that  $AC$  is to  $CB$  as  $AD$  to  $DB$ , so that  $A, B$  are one pair of conjugates and  $C, D$  the other.

First, let  $O$  be the middle point of  $CD$ , the line between one pair of conjugates.

Describe the circle on  $CD$  as diameter.

Take any point  $P$  on the circle, and draw  $PA, PC, PB, PO$ .



Because the angle  $OPC$  is equal to the angle  $OCP$ ,  
the sum of the angles  $OPB, BPC$  is equal to the sum of the angles  
 $CAP, CPA$ ; (I. Prop. 32.)

and because  $AP$  is to  $PB$  as  $AC$  to  $CB$ , (see page 427.)

the angle  $BPC$  is equal to the angle  $CPA$ ;

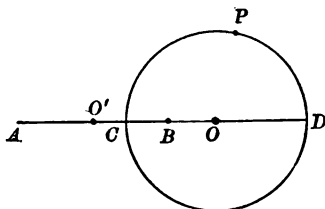
therefore the angle  $OPB$  is equal to the angle  $OAP$ ;

therefore  $OP$  touches the circle described about  $APB$ ;

(Converse of III. Prop. 32.)

therefore the square on  $OP$ , which is equal to the square on  $OC$ , is  
equal to the rectangle  $OA, OB$ . (III. Prop. 36, Coroll.)

Secondly, let  $O'$  be the middle point of  $AB$ , the line between the second pair of conjugates.



The rectangle  $OA, OB$  is equal to the difference between the squares on  $OO'$  and  $O'B$ , (II. Prop. 10.)

and the rectangle  $OA, OB$  is equal to the square on  $OC$ ;  
therefore the square on  $OC$  is equal to the difference between the squares on  $OO'$  and  $O'B$ .

Therefore the square on  $O'B$  is equal to the difference between the squares on  $OO'$  and  $OC$ , which is equal to the rectangle  $O'C, O'D$ .

(II. Prop. 10.)

**NOTE.** All the circles, which are the loci of the point  $P$  for different values of the ratio  $AP$  to  $BP$ , have their centres in the line  $AB$ , and, since the rectangle  $O'C, O'D$  is equal to the square on the tangent from  $O'$  to the circle  $CPD$ , the straight line which bisects  $AB$  at right angles is the radical axis of every pair of such circles. Such a series of circles is called **Coaxial**.

The points  $A, B$  are called the **limiting points** of the series of circles.

### EXERCISES.

1. If two circles be described upon the straight lines joining the two pairs of conjugate points of a harmonic range as diameters, the circles cut orthogonally.

2. A common tangent to two given circles is divided harmonically by any circle which is coaxial with the given circles.

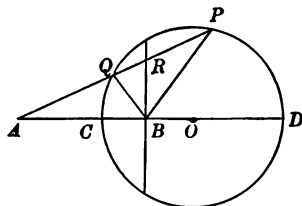
## ADDITIONAL PROPOSITION 7.

*A chord of a circle is divided harmonically by any point on it and the polar of the point.*

Let  $PQ$  be a chord of the given circle  $CQPD$ : take  $A$  any point on  $PQ$  produced and draw the diameter  $ACD$ .

Let  $B$  be the point such that  $ACBD$  is a harmonic range.

Draw  $PB$ ,  $BQ$  and draw  $BR$  at right angles to  $AB$  meeting  $PQ$  in  $R$ .



Because  $ACBD$  is a harmonic range and  $O$  is the middle point of  $CD$ , the rectangle  $OA, OB$  is equal to the square on  $OC$ ; (Add. Prop. 6.)

therefore  $BR$  is the polar of  $A$ . (see page 259.)

Because  $AP$  is to  $PB$  as  $AC$  to  $CB$ ,

and  $AQ$  is to  $QB$  as  $AC$  to  $CB$ ; (Add. Prop. 5.)

therefore  $AP$  is to  $PB$  as  $AQ$  to  $QB$ ;

and  $PB$  is to  $PA$  as  $QB$  to  $AQ$ ; (V. Def. 5 note.)

therefore  $PB$  is to  $BQ$  as  $PA$  to  $AQ$ ; (V. Prop. 9.)

therefore  $BA$  is the external bisector of the angle  $PBQ$ ;

(Prop. 3, Part 2.)

therefore  $BR$  is the internal bisector,

and  $PR$  is to  $RQ$  as  $PB$  to  $BQ$ , (Prop. 3, Part 1.)

and therefore as  $PA$  to  $AQ$ ;

therefore  $AQRP$  is a harmonic range.



It may be remarked that in the diagram on page 430 the point  $A$  is taken outside the circle. If the point were inside the circle, say  $R$ , its polar would intersect  $PQR$  in  $A$ . (Add. Prop. on page 262.)

Hence the theorem is established generally.

#### EXERCISES.

1. Prove that, if  $ACBD$  be a harmonic range, and if  $O$  be the middle point of  $CD$ , then  $AC$  is to  $CB$  as  $AO$  to  $OC$ .

2. Establish the theorem of page 426 by proving that in the figure of page 430 the triangles  $ABQ$ ,  $APO$  are similar and also that the triangles  $ABP$ ,  $AQO$  are similar.

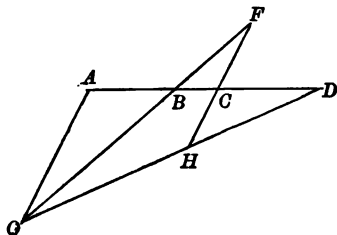
3. If any straight line  $PQR$  be drawn touching one given circle at  $Q$  and cutting another at  $P$ ,  $R$ , the segments  $PQ$ ,  $QR$  subtend equal or supplementary angles at either of the limiting points of the coaxial system to which the given circles belong.

4. If any straight line  $PQRS$  be drawn cutting two given circles of a coaxial system in  $P$ ,  $S$  and  $Q$ ,  $R$ , the segments  $PQ$ ,  $RS$  subtend equal or supplementary angles at either of the limiting points.

## ADDITIONAL PROPOSITION 8.

If a pencil be drawn from a point to the four points of a harmonic range and if a straight line be drawn through one of the points parallel to the ray which passes through the conjugate point, the part of the line intercepted between the rays through the other pair of points is bisected at the point.

Let  $ABCD$  be a harmonic range, and  $O$  be any point not in the straight line  $AD$ . Let  $OA$ ,  $OB$ ,  $OD$  be drawn\*, and let  $FCH$  be drawn parallel to  $AO$ , meeting  $OB$ ,  $OD$ , produced if necessary, in  $F$ ,  $H$ .



Because the triangles  $OAB$ ,  $FCB$  are equiangular to one another,  
 $OA$  is to  $FC$  as  $AB$  to  $BC$ ; (Prop. 4.)  
 and because the triangles  $OAD$ ,  $HCD$  are equiangular to one another,  
 $OA$  is to  $HC$  as  $AD$  to  $DC$ .

And because  $ABCD$  is a harmonic range,

$AB$  is to  $BC$  as  $AD$  to  $DC$ ;  
 therefore  $OA$  is to  $FC$  as  $OA$  to  $HC$ ; (V. Prop. 5.)  
 therefore  $FC$  is equal to  $HC$ . (V. Prop. 3.)

NOTE. The converse of this proposition is true, viz. if the line  $FH$  be bisected at  $C$ , then  $ABCD$  is a harmonic range.

## EXERCISE.

1. Give a construction to find the fourth point of a harmonic range when three points are given.

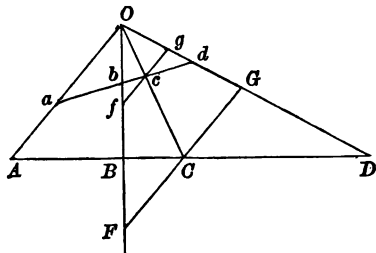
\* The ray  $OC$  of the pencil  $O(ABCD)$  is omitted in the figure, as it is not wanted in the proof. Similar omissions will be met with elsewhere.

## ADDITIONAL PROPOSITION 9.

*The points, in which a harmonic pencil is cut by any straight line, form a harmonic range.*

Let  $O(ABCD)$  be a pencil drawn through the points of the harmonic range  $ABCD$ : let  $abcd$  be any other straight line cutting the rays  $OA, OB, OC, OD$  in  $a, b, c, d$  respectively.

Through  $C, c$  draw  $GCF, gcf$  parallel to  $OA$  cutting the rays  $OB, OD$ , produced if necessary, in  $F, G$  and  $f, g$ .



Because  $ABCD$  is a harmonic range, and  $GCF$  is parallel to  $OA$ , therefore  $FC$  is equal to  $CG$ . (Add. Prop. 8.)

And because  $fcg$  is parallel to  $FCG$ , therefore  $fc$  is equal to  $cg$ . (Ex. 1, page 369.)

And because  $fcg$  is parallel to  $Oa$  and  $fc$  is equal to  $cg$ ,  $abcd$  is a harmonic range. (Note, p. 432.)

## EXERCISES.

1. The pencil formed by joining the four angular points of a square to any point on the circumscribing circle of the square is a harmonic pencil.

2. Give a construction for drawing the fourth ray of a harmonic pencil, when three rays are given.

3. Draw a harmonic pencil of which the rays pass through the angular points of a rectangle, and one of which is given in direction.

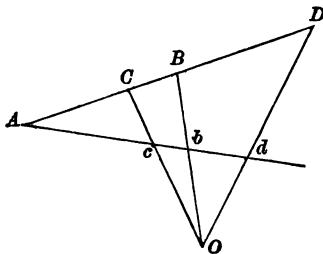
4.  $CA, CB$  are two tangents to a circle;  $E$  is the foot of the perpendicular from  $B$  on  $AD$  the diameter through  $A$ ; prove that  $CD$  bisects  $BE$ .

## ADDITIONAL PROPOSITION 10.

*If two harmonic ranges have two corresponding points, one in each range, coincident, the straight lines joining the other pairs of corresponding points pass through a point.*

Let  $ACBD$ ,  $Acbd$  be two harmonic ranges, of which the point  $A$  is a common point.

Draw  $Cc$ ,  $Bb$  and let them, produced if necessary, meet in  $O$ , and draw  $OD$ .



Because  $O(ACBD)$  is a harmonic pencil,

if  $Ac b$  cut  $OD$  in  $d'$ ,

then  $Ac b d'$  is a harmonic range; (Add. Prop. 9.)

therefore  $Ac$  is to  $cb$  as  $Ad'$  to  $d'b$ ;

but  $Ac b d$  is a harmonic range;

therefore  $Ac$  is to  $cb$  as  $Ad$  to  $db$ ;

therefore  $Ad'$  is to  $d'b$  as  $Ad$  to  $db$ , (V. Prop. 5.)

and  $d'$  coincides with  $d$ ; (Ex. 1, page 359.)

i.e.  $Cc$ ,  $Bb$ ,  $Dd$  meet in a point.

## EXERCISE.

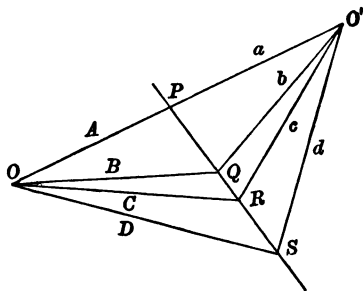
1. Prove that the intersections of the pairs of straight lines  $Cb$ ,  $Bc$ ;  $Bd$ ,  $Db$ ;  $Dc$ ,  $Cd$  in the above figure lie on a straight line which passes through  $A$ .

## ADDITIONAL PROPOSITION 11.

If two harmonic pencils have two corresponding rays, one of each pencil, coincident, the intersections of the other three pairs of corresponding rays lie on a straight line.

Let  $O(ABCD)$ ,  $O'(abcd)$  be two harmonic pencils, of which  $OaO'$  is a common ray.

Let  $OB$ ,  $O'b$  meet in  $Q$ , and  $OC$ ,  $O'c$  in  $R$ ; draw  $QR$  and let it meet  $OO'$  in  $P$ ,  $OD$  in  $S$ , and  $O'd$  in  $s$ .



Then because  $O(ABCD)$  is a harmonic pencil,

$PQRS$  is a harmonic range; (Add. Prop. 9.)

therefore  $PQ$  is to  $QR$  as  $PS$  to  $SR$ ;

and because  $O'(abcd)$  is a harmonic pencil,

$PQRs$  is a harmonic range;

therefore  $PQ$  is to  $QR$  as  $Ps$  to  $sR$ ;

therefore  $Ps$  is to  $sR$  as  $PS$  to  $SR$ , (V. Prop. 5.)

and the points  $S$ ,  $s$  coincide; (Ex. 1, page 359.)

i.e. the intersections of  $OB$ ,  $O'b$ ;  $OC$ ,  $O'c$ , and  $OD$ ,  $O'd$  are collinear.

## EXERCISE.

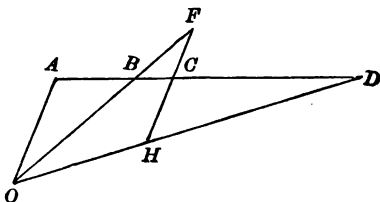
1. Prove that the straight lines joining the intersections of the pairs of straight lines  $OB$ ,  $O'c$  and  $OC$ ,  $O'b$ ;  $OC$ ,  $O'd$  and  $OD$ ,  $O'c$ ;  $OD$ ,  $O'b$  and  $OB$ ,  $O'd$  intersect in a point which lies on the line  $OO'$ .

## ADDITIONAL PROPOSITION 12.

*If a pencil be drawn from a point to the four points of an anharmonic range and if a straight line be drawn through one of the points parallel to the ray which passes through a second point, the part of it intercepted between the rays through the other pair of points will be divided in a constant ratio at the first point.*

Let  $ABCD$  be an anharmonic range and  $O$  be any point not in the straight line  $AD$ .

Let  $OA, OB, OD$  be drawn and  $FCH$  be drawn parallel to  $AO$  meeting  $OB, OD$ , produced if necessary, in  $F, H$ .



Because the triangles  $OAB, FCB$  are equiangular to one another,  
 $OA$  is to  $FC$  as  $AB$  to  $BC$ ; (Prop. 4.)  
 and because the triangles  $OAD, HCD$  are equiangular to one another,  
 $OA$  is to  $HC$  as  $AD$  to  $DC$ ;  
 therefore the ratio of the ratio  $OA$  to  $FC$  to the ratio  $OA$  to  $HC$  is  
 equal to the ratio of the ratio  $AB$  to  $BC$  to the ratio  $AD$  to  $DC$ ,  
 which is constant;  
 i.e. the ratio  $HC$  to  $FC$  is constant and is equal to the ratio of the  
 range  $ABCD$ . (Def. 10.)

## EXERCISES.

1. If  $ABCD, ABCE$  be two like anharmonic ranges, then the points  $D, E$  coincide.
2. If the ratio of the range  $ABCD$  be equal to the ratio of the range  $ADCB$ , the range  $ABCD$  is harmonic.

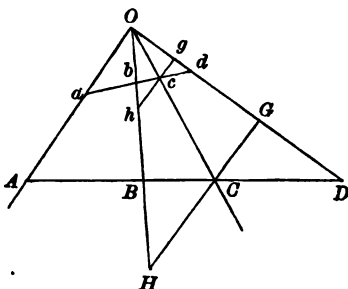
ADDITIONAL PROPOSITION 13.

The points in which an anharmonic pencil is cut by any straight line form an anharmonic range of constant ratio.

Let  $O(ABCD)$  be a pencil drawn through the points of the anharmonic range  $ABCD$ .

Let  $abcd$  be any other straight line cutting the rays  $OA$ ,  $OB$ ,  $OC$ ,  $OD$  in  $a$ ,  $b$ ,  $c$ ,  $d$  respectively.

Through  $C$ ,  $c$  draw  $GCH$ ,  $gch$  parallel to  $OA$  cutting the rays  $OD$ ,  $OB$  in  $G$ ,  $H$  and  $g$ ,  $h$ .



Because  $ABCD$  is an anharmonic range,  
and  $GCH$  is parallel to  $OA$ ,  
therefore the ratio of  $GC$  to  $CH$  is the ratio of the range  $ABCD$ ;  
(Add. Prop. 12.)

and because  $abcd$  is an anharmonic range and  $gch$  is parallel to  $Oa$ ,  
therefore the ratio of  $gc$  to  $ch$  is the ratio of the range  $abcd$ ;  
(Add. Prop. 12.)

and because  $gch$  is parallel to  $GCH$ ,  
 $gc$  is to  $ch$  as  $GC$  to  $CH$ ; (Ex. 2, page 365.)  
therefore  $abcd$  is an anharmonic range of ratio equal to that of the  
range  $ABCD$ .

EXERCISES.

1. Find a point on a given straight line such that lines drawn from it to three given points shall intercept on any parallel to the given line lengths having a given ratio.

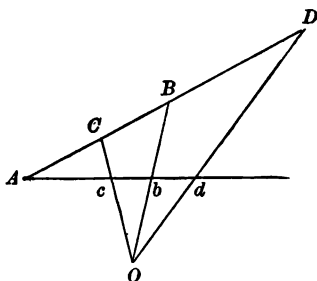
2. Three points  $F$ ,  $G$ ,  $H$  are taken on the side  $BC$  of a triangle  $ABC$ : through  $G$  any line is drawn cutting  $AB$  and  $AC$  in  $L$  and  $M$  respectively;  $FL$  and  $HM$  intersect in  $K$ ; prove that  $K$  lies on a fixed straight line passing through  $A$ .

#### ADDITIONAL PROPOSITION 14.

*If two like anharmonic ranges have two corresponding points, one in each range, coincident, the straight lines joining the other pairs of corresponding points pass through a point.*

Let  $ACBD$ ,  $Acbd$ , be two like anharmonic ranges, of which the point  $A$  is a common point.

Draw  $Cc$ ,  $Bb$ , and let them, produced if necessary, meet in  $O$ ; and draw  $OD$ .



Because  $O(ACBD)$  is an anharmonic pencil,  
if  $Acb$  cut  $OD$  in  $d'$ ,  
then  $Acb'd'$  is an anharmonic range of ratio equal to that of the pencil;  
(Add. Prop. 13.)

and  $Acdb$  is a like anharmonic range;  
therefore the ratio of the ratio  $Ac$  to  $cb$  to the ratio  $Ad'$  to  $d'b$  is equal  
to the ratio of the ratio  $Ac$  to  $cb$  to the ratio  $Ad$  to  $db$ ;  
therefore the ratio  $Ad'$  to  $d'b$  is equal to the ratio  $Ad$  to  $db$ ,  
and  $d'$  coincides with  $d$ , (Ex. 1, page 359.)  
i.e.  $Cc$ ,  $Bb$ ,  $Dd$  meet in a point.

### EXERCISE.

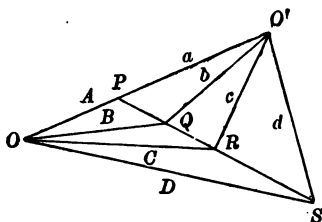
1. Prove that the intersections of the pairs of straight lines  $Cb, Bc; Bd, Db; Dc, Cd$  in the above figure lie on a straight line through  $A$ .



## ADDITIONAL PROPOSITION 15.

If two like anharmonic pencils have two corresponding rays, one in each pencil, coincident, the intersections of the other three pairs of corresponding rays lie on a straight line.

Let  $O(ABCD)$ ,  $O'(abcd)$  be two like anharmonic pencils, of which  $OAaO'$  is a common ray. Let  $OB$ ,  $O'b$  meet in  $Q$ , and  $OC$ ,  $O'c$  in  $R$ ; draw  $QR$  and let it meet  $OO'$  in  $P$ ,  $OD$  in  $S$  and  $O'd$  in  $s$ .



Because  $PQRS$  is a transversal of the anharmonic pencil  $O(ABCD)$ ,  $PQRS$  is a range of ratio equal to that of the pencil; (Add. Prop. 13) and because  $PQRs$  is a transversal of the anharmonic pencil  $O'(abcd)$ ,

$PQRs$  is a range of ratio equal to that of the pencil;

and because  $O(ABCD)$ ,  $O'(abcd)$  are like anharmonic pencils,

(Hypothesis)

therefore  $PQRS$ ,  $PQRs$  are two like anharmonic ranges;

therefore the points  $S$ ,  $s$  coincide; (Ex. 1, page 436)

i.e. the intersections of  $OB$ ,  $O'b$ ;  $OC$ ,  $O'c$ ; and  $OD$ ,  $O'd$  are collinear.

## EXERCISE.

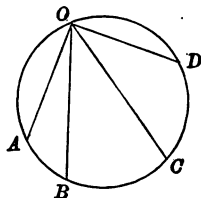
1. Prove that the straight lines joining the intersections of the pairs of straight lines  $OB$ ,  $O'c$  and  $OC$ ,  $O'b$ ;  $OC$ ,  $O'd$  and  $OD$ ,  $O'c$ ;  $OD$ ,  $O'b$  and  $OB$ ,  $O'd$  intersect in a point which lies on the straight line  $OO'$ .

## ADDITIONAL PROPOSITION 16.

*The anharmonic ratio of the pencil formed by joining four given points on a circle to any fifth point on the same circle is constant.*

Let  $A, B, C, D$  be four given points on a circle, and let  $O$  be any fifth point on the circle, and let the pencil  $O(ABCD)$  be drawn.

Take  $O'$  any other point in the same arc  $AD$  as  $O$ ;  
then the angles  $AO'B, BO'C, CO'D$  are equal to the angles  $AOB, BOC, COD$  respectively;  
and the pencil  $O'(ABCD)$  is equal \* to the pencil  $O(ABCD)$ .



Next take  $O'$  any point in the arc  $AB$ .

Then the angles  $BO'C, CO'D$  are equal to the angles  $BOC, COD$  respectively,

and the angle between  $O'B$  and  $AO'$  produced, say  $O'A_1$ , is equal to the angle  $AOB$ ,

and the pencil  $O'(A_1BCD)$  is equal to the pencil  $O(ABCD)$ .

Similarly it can be proved that the pencil is the same for all positions of  $O'$  on the circle.

## EXERCISE.

1. The locus of the vertex of a harmonic pencil, whose rays pass through the angular points of a square, is the circumscribed circle of the square.

\* In the sense that one pencil can be shifted so that its rays coincide with the rays of the other pencil. (I. Def. 21.)

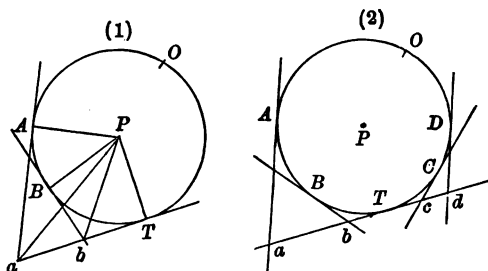
ADDITIONAL PROPOSITION 17.

*The anharmonic ratio of the range formed by the intersections of four given tangents to a circle by any fifth tangent to the same circle is constant.*

Let  $Aa, Bb, Cc, Dd$  be the tangents at four given points,  $A, B, C, D$  on a circle, and let them cut the tangent at any fifth point  $T$  on the circle in  $a, b, c, d$ .

Find  $P$  the centre, and take any point  $O$  on the circle.

Draw  $PA, PB, PT, Pa, Pb$ .



Because the angle  $APT$  is equal to twice the angle  $AOT$ ,

(III. Prop. 20)

and also equal to twice the angle  $aPT$ ,

the angle  $AOT$  is equal to the angle  $aPT$ .

Similarly it can be proved that

the angle  $BOT$  is equal to the angle  $bPT$ ;

therefore the angle  $AOB$  is equal to the angle  $aPb$ .

Similarly it can be proved that the angles  $BOC, COD$  are equal to the angles  $bPc, cPd$ ;

therefore the pencil  $O(ABCD)$  is equal to the pencil  $P(abcd)$ ,

and the pencil  $O(ABCD)$  has a constant ratio; (Add. Prop. 16)

therefore the pencil  $P(abcd)$  has a constant ratio;

and therefore the range  $abcd$  has a constant ratio. (Add. Prop. 13)

EXERCISE.

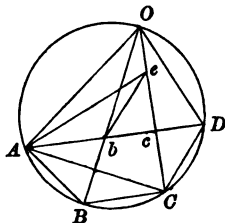
1. If a straight line cut the four sides of a square in a harmonic range, it touches the inscribed circle of the square.

## ADDITIONAL PROPOSITION 18.

*The anharmonic ratio of the pencil formed by joining four points on a circle to any fifth point on the circle is the same as the ratio of the rectangles contained by the chords that join the points.*

Let  $A, B, C, D$  be four given points on a circle. Draw  $AB, AC, AD, BC, CD$ .

In  $AD$  take  $Ab$  equal to  $AB$ ; draw  $Bb$  and let it be produced to meet the circle in  $O$ ; draw  $OA, OC, OD$  and let  $OC$  cut  $AD$  in  $c$ ; draw  $be$  parallel to  $CD$  to meet  $OC$  in  $e$ , and draw  $Ae$ .



Because  $be$  is parallel to  $CD$ ,

the angle  $bec$  is equal to the angle  $OCD$ ,  
which is equal to the angle  $OAc$ ; (III. Prop. 21)  
therefore  $A, b, e, O$  lie on a circle.

Therefore the angle  $Aeb$  is equal to the angle  $AOB$ ,  
which is equal to the angle  $ACB$ ;

and the angle  $Abe$  is equal to the supplement of the angle  $AOe$ ,  
which supplement is equal to the angle  $ABC$ ;  
therefore the triangles  $Abe, ABC$  are equiangular to one another;  
and  $Ab$  is equal to  $AB$ ; (Constr.)  
therefore  $be$  is equal to  $BC$ .

Now the anharmonic ratio of the pencil  $O(ABCD)$  is equal to that of the range  $AbcD$ , (Add. Prop. 13)  
which is equal to the ratio of the ratio  $Ab$  to  $bc$  to the ratio  $AD$  to  $Dc$ ,  
that is, to the ratio compounded of the ratios  $Ab$  to  $bc$  and  $Dc$  to  $AD$ ,  
which is the ratio of the rectangle  $Ab, cD$  to the rectangle  $AD, bc$ .  
(See page 399.)

And because  $be$  is parallel to  $CD$ ,

the triangles  $DcC$ ,  $bce$  are equiangular to one another ;

therefore  $cD$  is to  $bc$  as  $CD$  to  $be$  ; (Prop. 4)

therefore the ratio of the rectangle  $Ab$ ,  $cD$  to the rectangle  $AD$ ,  $bc$  is equal to the ratio of the rectangle  $AB$ ,  $CD$  to the rectangle  $AD$ ,  $be$ .

Therefore the anharmonic ratio of the pencil  $O$  ( $ABCD$ ) is equal to the ratio of the rectangle  $AB$ ,  $CD$  to the rectangle  $AD$ ,  $BC$ .

The anharmonic ratio of a range  $ABCD$  is defined to be (Definition 10) the ratio of the ratio  $AB$  to  $BC$  to the ratio  $AD$  to  $DC$ , which, by Definition 8 of Book V., is equal to the ratio compounded of the ratios  $AB$  to  $BC$  and  $DC$  to  $AD$  ; and this last ratio has, on page 399, been shewn to be equal to the ratio of the rectangle  $AB$ ,  $CD$  to the rectangle  $AD$ ,  $BC$ .

Now it can be proved (Ex. 1, page 137) that, if  $ABCD$  be a range, the sum of the rectangles  $AB$ ,  $CD$  and  $AD$ ,  $BC$  is equal to the rectangle  $AC$ ,  $BD$ .

If therefore any two of these three rectangles be given, the third is at once found. There are six ratios, of which one of these rectangles is the antecedent and another the consequent ; if any one of these ratios be given, the other five ratios are at once found. Now in the definition the ratio of the rectangle  $AB$ ,  $CD$  to the rectangle  $AD$ ,  $BC$  is defined as *the* anharmonic ratio of the range  $ABCD$ . There is no reason against adopting any other of the six ratios as *the* ratio of the range, but it is important strictly to adhere throughout an investigation to one and the same ratio.

### EXERCISES.

1. The anharmonic ratio of the range formed by the intersections of four given tangents to a circle with any fifth tangent is equal to the ratio of the rectangles contained by the chords which join the points of contact of the given tangents.

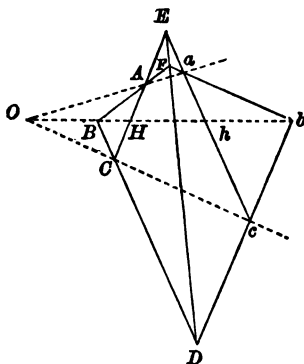
2. Two fixed points  $D$ ,  $E$  are taken on the diameter  $AB$  of a circle, and  $P$  any point on the circumference ; perpendiculars  $AM$ ,  $BN$  are let fall on  $PD$ ,  $PE$  ; prove that the ratio of the rectangle  $PM$ ,  $PN$  to the rectangle  $AM$ ,  $BN$  is constant.

## ADDITIONAL PROPOSITION 19.

If two triangles be such that the straight lines joining their vertices in pairs pass through a point, the intersections of pairs of corresponding sides lie on a straight line.

Let  $ABC, abc$  be two given triangles such that the straight lines  $Aa, Bb, Cc$  meet in a point  $O$ .

Let the pairs of sides  $BC, bc; CA, ca; AB, ab$ , produced if necessary, meet in  $D, E, F$  respectively, and let  $OBb$  cut  $AC, ac$  in  $H, h$ .



Because  $EAHC, Eahc$  cut the same pencil  $O(EAHC)$ ,  
 $EAHC, Eahc$  are like anharmonic ranges; (Add. Prop. 13),  
 therefore the pencils  $B(EAHC), b(Eahc)$  are like anharmonic pencils;  
 and they have a common ray  $BHhb$ ;  
 therefore the intersections of the pairs of rays  $BC, bc; BE, bE$ ;  
 $BA, ba$  lie on a straight line; (Add. Prop. 15);  
 that is,  $D, E, F$  lie on a straight line.

NOTE. The point  $O$  is often called the **pole** of the triangles  $ABC, abc$ , and the straight line  $DEF$  the **axis** of the triangles.

This theorem may then be enunciated thus, *compolar triangles are coaxial*.

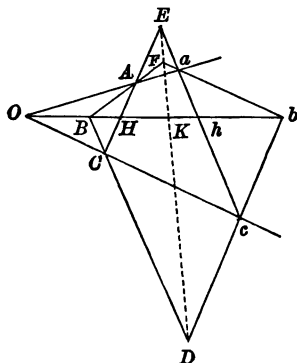
Compolar triangles are often said to be **in perspective**.

## ADDITIONAL PROPOSITION 20.

If two triangles be such that the intersections of their sides taken in pairs lie on a straight line, the straight lines joining pairs of corresponding vertices meet in a point.\*

Let  $ABC$ ,  $abc$  be two given triangles such that the pairs of sides  $BC$ ,  $bc$ ;  $CA$ ,  $ca$ ;  $AB$ ,  $ab$  intersect in three points  $D$ ,  $E$ ,  $F$  lying on a straight line.

Let the straight lines  $Aa$ ,  $Bb$ ,  $Cc$  be drawn and let  $Bb$  cut  $AC$ ,  $ac$ ,  $DEF$  in  $H$ ,  $h$ ,  $K$  respectively.



Because the pencils  $B(EFKD)$ ,  $b(EFKD)$  have a common transversal  $EFKD$ ,

they are like anharmonic pencils;

and because  $EAHC$  is a transversal of the pencil  $B(EFKD)$ ,

and  $Eahc$  is a transversal of the pencil  $b(EFKD)$ ,

$EAHC$ ,  $Eahc$  are like anharmonic ranges; (Add. Prop. 13)

and they have a common point  $E$ ;

therefore the lines  $Aa$ ,  $Hh$ ,  $Cc$  meet in a point, (Add. Prop. 14)

that is,  $Aa$ ,  $Bb$ ,  $Cc$  meet in a point.

NOTE. The above theorem may be enunciated thus, *coaxial triangles are compolar*.

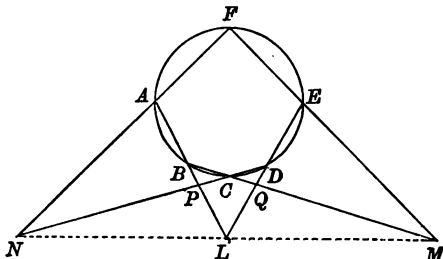
\* The theorems of this and the preceding pages are attributed to Gerard Desargues (born at Lyons 1593, died 1662).

## ADDITIONAL PROPOSITION 21.

*If a hexagon be inscribed in a circle, the intersections of pairs of opposite sides lie on a straight line.\**

Let  $ABCDEF$  be a hexagon inscribed in a given circle.

Let the pairs of sides  $AB, DE$ ;  $BC, EF$ ;  $CD, FA$  meet in  $L, M, N$  respectively, and let  $AB, CD$  meet in  $P$ , and  $BC, DE$  in  $Q$ .



Because  $A, B, C, D, E, F$  are points on a circle,  
 $A(BCDF), E(BCDF)$  are like anharmonic pencils; (Add. Prop. 16)  
 and because  $PCDN$  is a transversal of the pencil  $A(BCDF)$ ,  
 and  $BCQM$  is a transversal of the pencil  $E(BCDF)$ ,  
 $PCDN, BCQM$  are like anharmonic ranges; (Add. Prop. 13)  
 and they have a common point  $C$ ;  
 therefore the lines  $PB, DQ, NM$  meet in a point, (Add. Prop. 14)  
 that is,  $AB, DE, NM$  meet in a point,  
 or, in other words, the points  $L, M, N$  lie on a straight line.

## EXERCISES.

1. If the tangents to the circumscribed circle of a triangle  $ABC$  at  $A, B, C$  meet the sides  $BC, CA, AB$ , in  $L, M, N$ , then  $L, M, N$  lie on a straight line.

2. If  $ACE, BDF$  be two ranges, then the intersections of the pairs of straight lines  $AB, DE$ ;  $BC, EF$ ;  $CD, FA$  are collinear.

\* This theorem is true for any conic. It was discovered at the age of sixteen by Blaise Pascal (born at Clermont 1623, died at Paris, 1662).

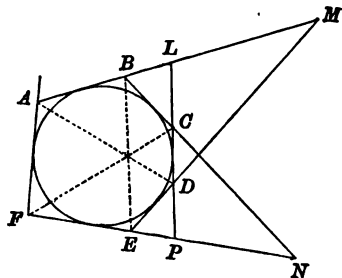


ADDITIONAL PROPOSITION 22.

*If a hexagon be described about a circle, the straight lines joining opposite vertices pass through a point.\**

Let  $ABCDEF$  be a hexagon described about a given circle.

Let  $AB$  meet  $CD$ ,  $DE$  in  $L$ ,  $M$  and  $EF$  meet  $BC$ ,  $CD$  in  $N$ ,  $P$ .



Because the six sides of the hexagon are tangents to a circle, the points where the sides  $AB$ ,  $EF$  are cut by the remaining sides of the hexagon form like anharmonic ranges, (Add. Prop. 17)

that is,  $ABLM$ ,  $FNPE$  are like anharmonic ranges;

therefore  $D(ABLM)$ ,  $C(FNPE)$  are like anharmonic pencils;

and they have a common ray  $LCDP$ ;

therefore the pairs of rays  $DA$ ,  $CF$ ;  $DB$ ,  $CN$ ;  $DM$ ,  $CE$  intersect on a straight line, (Add. Prop. 15)

that is,  $DA$ ,  $CF$  meet in a point on the line  $BE$ ,

or, in other words,  $AD$ ,  $BE$ ,  $CF$  meet in a point.

EXERCISE.

1. If the inscribed circle of a triangle  $ABC$  touch the sides  $BC$ ,  $CA$ ,  $AB$  in  $D$ ,  $E$ ,  $F$ , then  $AD$ ,  $BE$ ,  $CF$  meet in a point.

\* This theorem also is true for any conic. It was discovered by Charles Julien Brianchon (born at Sèvres, 1785).

## SIMILAR FIGURES.

If two similar figures be placed so that their corresponding sides are parallel in pairs, the figures are then said to be **similar and similarly situate**.

It appears from Proposition 32 that, if two triangles be similar and similarly situate, then the lines joining pairs of corresponding vertices meet in a point. It is easily proved that, if two polygons of any number of sides be similar and similarly situate, then the lines joining pairs of corresponding vertices meet in a point, and further that the ratio of the distances of this point from any pair of corresponding points of the two figures is constant, and is equal to the ratio of a pair of corresponding sides of the two figures. Such a point is called in consequence a **centre of similitude** of the two figures.

It is also easily seen that, if a point be a centre of similitude of two figures, it is also a centre of similitude of any two figures similarly described with reference to the first pair of similar figures; for instance, if a point be a centre of similitude of two triangles, it is also a centre of similitude of the circumscribed circles of the triangles, and also a centre of similitude of the inscribed circles and so on.

If a pair of corresponding points lie on the same side of the centre of similitude, it is called a **centre of direct similitude**: if a pair of corresponding points lie on opposite sides of the centre of similitude, it is called a **centre of inverse similitude**.

It must be noticed that the centre of direct similitude is at an infinite distance in the case when the two similar figures are equal.

As an example we will prove an important theorem with reference to the centres of similitude of the circumscribed circle and the Nine Point circle of a triangle:

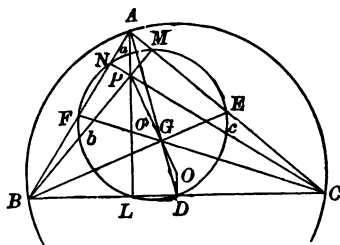
*The orthocentre and the centroid of a triangle are the centres of direct and inverse similitude of the circumscribed circle and the Nine Point circle of the triangle.*

If  $ABC$  be a triangle, and  $D, E, F$  be the middle points of its sides, the triangles  $ABC, DEF$  are equiangular to one another, and the ratio of  $AB$  to  $DE$  is equal to the ratio 2 to 1.

(Add Prop., page 101.)

Also the radii of the circles  $ABC, DEF$  are in the ratio of 2 to 1.

Because the lines  $AD, BE, CF$  meet in a point  $G$ , and the ratios  $AG$  to  $GD, BG$  to  $GE, CG$  to  $GF$  are each equal to 2 to 1, (Add Prop., page 103.)



$G$  is the centre of inverse similitude of the two circles  $ABC, DEF$ .

Again, if  $P$  be the orthocentre of the triangle  $ABC$ ,

and  $a, b, c$  be the middle points of  $PA, PB, PC$ ,

the ratios  $PA$  to  $Pa, PB$  to  $Pb, PC$  to  $Pc$  are each equal to 2 to 1.

Therefore  $P$  is the centre of direct similitude of the triangles  $ABC, abc$ , and therefore of the circles  $ABC, abc$ .

Now the circles  $abc, DEF$  are identical. (Add. Prop., page 271.)

Therefore  $P$  is the centre of direct similitude of the circles  $ABC, DEF$ .

Now, if  $O, O'$  be the centres of the circles  $ABC, DEF$ ,

since the radii of the circles are in the ratio 2 to 1,

their centres of similitude lie in the straight line  $OO'$ ,

and divide the distance internally and externally in the ratio 2 to 1.

That is,  $P, O', G, O$  lie on a straight line,

and  $PO$  is equal to twice  $PO'$ ,

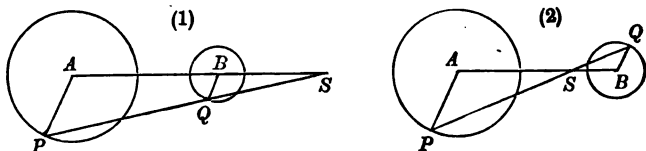
and  $GO$  is equal to twice  $GO'$ .

## ADDITIONAL PROPOSITION 23.

*Every straight line which passes through the extremities of two parallel radii of two fixed circles passes through a centre of similitude of the circles.*

Let  $AP$ ,  $BQ$  be two parallel radii of two given circles, whose centres are  $A$ ,  $B$ .

Draw  $AB$ ,  $PQ$  and let them, produced if necessary, meet at  $S$ .



Because the triangles  $APS$ ,  $BQS$  are equiangular to one another,

$AS$  is to  $BS$  as  $AP$  to  $BQ$ ; (Prop. 4)

that is,  $S$  is a point, which divides the distance  $AB$  externally (fig. 1) or internally (fig. 2) in the ratio of the radii;

therefore  $S$  is a fixed point. (Ex. 1, page 359.)

Again, because the triangles  $APS$ ,  $BQS$  are equiangular to one another,  $SP$  is to  $SQ$  as  $AP$  to  $BQ$ ; (Prop. 4)

that is, the ratio of the distances  $SP$  to  $SQ$  is a constant ratio;  
therefore  $S$  is a centre of similitude of the circles.

In figure 1, where the two radii  $AP$ ,  $BQ$  are drawn in the same sense,  $S$  is in the line of centres  $AB$  produced, and the two distances  $SP$ ,  $SQ$  are drawn in the same direction.  $S$  is a centre of direct similitude.

In figure 2, where  $AP$ ,  $BQ$  are drawn in opposite senses,  $S$  is in the line of centres  $AB$ , and the two distances  $SP$ ,  $SQ$  are drawn in opposite directions.  $S$  is a centre of inverse similitude.

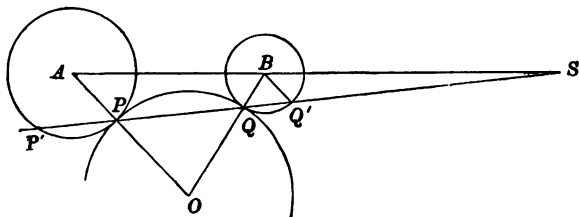
## ADDITIONAL PROPOSITION 24.

*If a circle be drawn to touch two given circles, the straight line which passes through the points of contact passes through one of the centres of similitude of the given circles.*

Let a circle be drawn to touch two given circles, whose centres are  $A, B$ , at  $P, Q$ .

Draw  $AP, BQ$  and let them, produced if necessary, meet at  $O^*$ .

Draw  $PQ$  and let it, produced if necessary, meet the circles again at  $P', Q'$ : draw  $BQ'$ .



Because the circles touch at  $P$ , the centre of the circle which is described to touch the given circles must lie in  $AP$ ;

(III. Prop. 10, Coroll.)

similarly it must lie in  $BQ$ ;

therefore  $O$  is the centre;

therefore the angle  $OPQ$  is equal to the angle  $OQP$ ,

which is equal to the angle  $BQQ'$  and therefore to the angle  $BQ'Q$ ;

therefore  $APQ, BQ'$  are parallel; (I. Prop. 28)

therefore  $PQ$  passes through a centre of similitude. (Add. Prop. 23.)

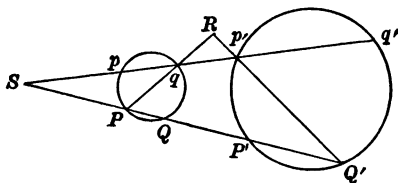
\* If  $AP, BQ$  be parallel, so that the point  $O$  is infinitely distant, and the radius of the circle which touches the given circles at  $P, Q$  is infinitely large,  $PQ$  becomes one of the common tangents of the given circles, which common tangents pass through  $S$ .

## ADDITIONAL PROPOSITION 25.

*If two straight lines be drawn through a centre of similitude of two given circles to cut the circles, a pair of chords of the two circles joining pairs of inverse\* points intersect on the radical axis of the given circles, and a pair of chords of the two circles joining pairs of corresponding points are parallel.*

Let  $S$  be a centre of similitude of two given circles  $PQqp$ ,  $P'Q'q'p'$  and let  $SPQP'Q'$ ,  $Spqp'q'$  be any two straight lines drawn through  $S$  cutting the circles;  $P$ ,  $Q'$ ;  $Q$ ,  $P'$ ;  $p$ ,  $q'$ ;  $q$ ,  $p'$  being pairs of inverse points and  $P$ ,  $P'$ ;  $Q$ ,  $Q'$ ;  $p$ ,  $p'$ ;  $q$ ,  $q'$  being pairs of corresponding points.

Draw  $Pq$ ,  $Q'p'$ , and let them, produced if necessary, meet at  $R$ .



Because  $S$  is a centre of similitude,  $Sp$  is to  $Sp'$  as  $SQ$  to  $SQ'$ ;

therefore  $Qp$ ,  $Q'p'$  are parallel; (Prop. 2, Part 2)

therefore the angles  $PQp$ ,  $PQ'p'$  are equal; (I. Prop. 29)

and the angles  $PQp$ ,  $Pqp$  are equal; (III. Prop. 21)

therefore the angles  $PQ'p'$ ,  $Pqp$  are equal.

Therefore the four points  $P$ ,  $q$ ,  $p'$ ,  $Q'$  lie on a circle,

(III. Prop. 22, Coroll.)

and the rectangle  $RP$ ,  $Rq$  is equal to the rectangle  $RQ'$ ,  $Rp'$ ;

(III. Prop. 36)

therefore the squares on the tangents drawn from  $R$  to the two circles are equal,

and consequently  $R$  lies on the radical axis of the given circles.

(Add. Prop. page 264.)

\* For this name see page 460.

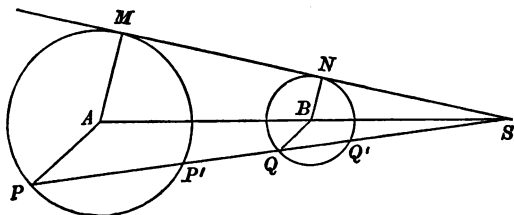
## ADDITIONAL PROPOSITION 26.

*If a straight line be drawn through a centre of similitude of two given circles, the rectangle contained by the distances of two inverse points is constant.*

Let  $S$  be a centre of similitude of two given circles, whose centres are  $A, B$ .

Draw  $MNS$  a common tangent through  $S$  (see note on page 451), and let  $SQ'QP'P$  be any straight line through  $S$  cutting the circles in  $P, P'$  and  $Q, Q'$ .

Draw  $AP, BQ, AM, BN$ .



Because the rectangle  $SP, SP'$  is equal to the square on  $SM$ ,

$SP$  is to  $SM$  as  $SM$  to  $SP'$ . (Prop. 17, Part 2.)

And because  $S$  is a centre of similitude,

$SP$  is to  $SQ$  as  $SM$  to  $SN$ ;

and therefore  $SP$  is to  $SM$  as  $SQ$  to  $SN$ ; (V. Prop. 9)

therefore  $SQ$  is to  $SN$  as  $SM$  to  $SP'$ ; (V. Prop. 5)

therefore the rectangle  $SQ, SP'$  is equal to the rectangle  $SM, SN$ .

(Prop. 16, Part 1.)

Similarly it can be proved that the rectangle  $SP, SQ'$  is equal to the rectangle  $SM, SN$ .

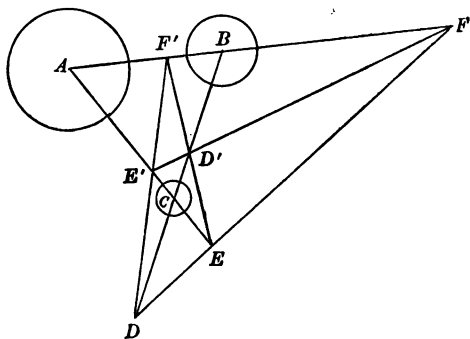
## EXERCISE.

1. Prove that in the above figure the rectangle  $PQ, P'Q'$  is equal to the square on  $MN$ .

## ADDITIONAL PROPOSITION 27.

*The six centres of similitude of three given circles taken in pairs lie three by three on four straight lines\*.*

Let  $A, B, C$  be the centres of three given circles; let  $a, b, c$  be the radii of the  $A, B, C$  circles, and let  $D, D'$  be the centres of direct and of inverse similitude of the  $B, C$  circles,  $E, E'$  those of the  $C, A$  circles, and  $F, F'$  those of the  $A, B$  circles.



Because  $BD$  is to  $DC$  as  $b$  to  $c$ ,  
 and  $CE$  is to  $EA$  as  $c$  to  $a$ ,  
 and  $AF$  is to  $FB$  as  $a$  to  $b$ , (Add. Prop. 23)

therefore the ratio compounded of the ratios

$BD$  to  $DC$ ,  $CE$  to  $EA$  and  $AF$  to  $FB$ ,

is equal to the ratio compounded of the ratios

$b$  to  $c$ ,  $c$  to  $a$  and  $a$  to  $b$ , that is to unity;

therefore  $DEF$  is a straight line. (Add. Prop. 2.)

Similarly it can be proved that  $DE'F'$ ,  $D'E'F$ ,  $D'E'F'$  are straight lines,

and also that the lines of each of the sets  $AD', BE', CF'$ ;  $AD, BE, CF$ ;  $AD, BE', CF'$  meet in a point. (Add. Prop. 4.)

\* These lines are called the axes of similitude of the three circles.

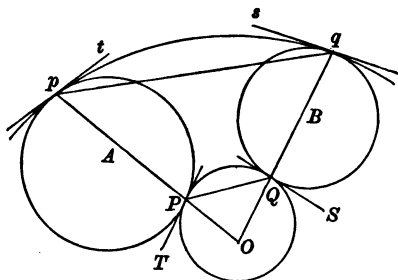


## ADDITIONAL PROPOSITION 28.

If a point on the radical axis of two given circles be joined to the points of contact of a circle, which touches both the given circles, by straight lines which cut the circles again, another circle can be described to touch the given circles at the points of section.

Let  $O$  be a point on the radical axis of the given circles  $A, B$ ; and let a circle touch them at  $P, Q$ .

Draw  $OP, OQ$  and let them meet the  $A, B$  circles in  $p, q$ . Draw  $PT, QS, pt, qs$  the tangents to the circles at  $P, Q, p, q$  and draw  $PQ, pq$ .



Because  $O$  is on the radical axis of the circles  $A, B$ ,  
the rectangle  $OP, Op$  is equal to the rectangle  $OQ, Oq$ ;  
therefore  $P, Q, q, p$  lie on a circle. (Ex. 1, page 253.)

The difference of the angles  $tpO, sqO$  is equal to  
the difference of the angles  $TPO, SQO$ ,  
which is equal to the difference of the angles  $PQO, QPO$ ,  
which is equal to the difference of the angles  $qpO, pqO$ ;  
(III. Prop. 22)

therefore the angle  $tpq$  is equal to the angle  $sqp$ .

It is therefore possible to describe a circle touching the circles  $A, B$  at  $p, q$ .

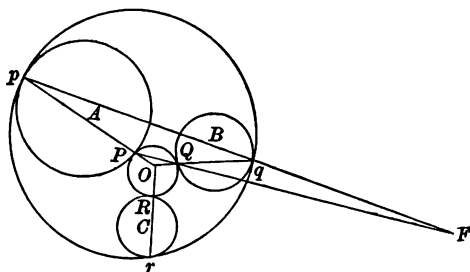
**COROLLARY.** If the radical centre of three given circles be joined to the points of contact of a circle, which touches all the three given circles, by straight lines which cut the circles again, another circle can be described to touch the given circles at the points of section.

(See Ex. 140, page 284.)

## ADDITIONAL PROPOSITION 29.

*If two circles be drawn to touch three given circles, so that the straight line joining the two points of contact on each of the given circles passes through the radical centre of the given circles, the radical axis of the pair of circles is one of the axes of similitude of the three given circles.*

Let  $PQR, pqr$  be a pair of circles touching three given circles  $A, B, C$  at  $P, p; Q, q; R, r$ , so that  $Pp, Qq, Rr$  pass through  $O$  the radical centre of the circles  $A, B, C$ . (Add. Prop. 28, Coroll.)



Draw  $PQ, pq$  and let them, produced if necessary, meet in  $F$ .

Because the circles  $PQR, pqr$  touch the circles  $A, B$ , the lines  $PQ, pq$  pass through one of the centres of similitude of the circles  $A$  and  $B$ ; (Add. Prop. 24)

therefore the rectangle  $FP, FQ$  is equal to the rectangle  $Fp, Fq$ ; (Add. Prop. 26.)

therefore  $F$  is a point on the radical axis of the circles  $PQR, pqr$ .

Similarly it can be proved that a centre of similitude of each of the pairs of circles  $B, C$  and  $C, A$  lies on the radical axis of  $PQR, pqr$ .

Therefore the radical axis of the circles  $PQR, pqr$  is an axis of similitude of the circles  $A, B, C$ .

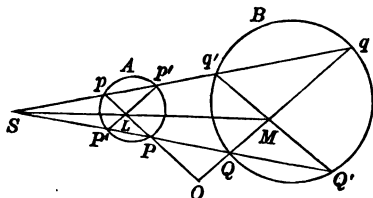
## ADDITIONAL PROPOSITION 80.

*If a pair of chords of two given circles intersect in the radical axis, and if the chords intersect the polars of one of the centres of similitude in two points collinear with the centre of similitude, then the points of intersection of the chords with the circles lie two by two on two straight lines through the centre of similitude.*

Let  $S$  be one of the centres of similitude of two given circles  $A, B$ ; let  $L, M$  be two points collinear with  $S$ , on the polars of  $S$  with respect to the circles  $A, B$ ; and let  $O$  be a point on the radical axis.

Draw  $OL$ , and let it intersect the circle  $A$  in  $P, p$ .

Draw  $SP, Sp$ , and let them meet the circle  $A$  in  $P', p'$ , and let the corresponding points to  $P, P', p, p'$ , where the lines meet the circle  $B$  be  $Q, Q', q, q'$ .



Because  $L$  is on the polar of  $S$ , the line  $P'p'$  passes through  $L$ .

(Add. Prop. page 261.)

Because  $S$  is the centre of similitude, and the intersections of  $Pp, P'p'$ , and of  $Qq, Q'q'$  are corresponding points, and  $Pp, P'p'$  intersect at  $L$ ,

and  $L, M$  are corresponding points,

therefore  $Qq, Q'q'$  intersect at  $M$ ;

and because  $S$  is the centre of similitude,

$Pp, Qq$  intersect on the radical axis, (Add. Prop. 25)

that is,  $Qq$  must pass through  $O$ ; and it also passes through  $M$ .

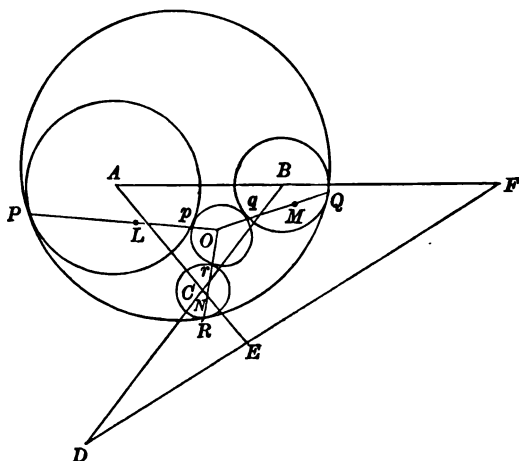
Therefore the points  $Q, q$  obtained by the above construction are the points where the straight line  $OM$  cuts the circle  $B$ , that is, if  $OL$  cut the circle  $A$  in  $P, p$  and  $OM$  cut the circle  $B$  in  $Q, q$ , then  $PQ, pq$  pass through  $S$ .

## ADDITIONAL PROPOSITION 31.

*To describe a circle to touch three given circles.\**

Let  $A, B, C$  be three given circles. Find  $O$  the radical centre of the circles  $A, B, C$ , and draw  $DEF$  one of their axes of similitude.

Find the poles  $L, M, N$  of the line  $DEF$  with respect to the circles  $A, B, C$  respectively. Draw  $OL, OM, ON$  and let them, produced if necessary, cut the circles  $A, B, C$  respectively in  $P, p; Q, q; R, r$ .



Because  $L, M$  are the poles of  $DEF$  with respect to the circles  $A, B$ , therefore  $L, M$  are on the polars of  $F$  with respect to the circles  $A, B$ ;

and because  $F$  is a centre of similitude of the two circles, therefore  $LMF$  is a straight line.

Therefore the line  $PQ$  passes through  $F$ . (Add. Prop. 30.)

\* This solution of the problem is due to Joseph Diez Gergonne (born at Nancy 1771, died at Montpellier 1859).

Therefore a circle can be described through  $P, Q$  to touch the given circles at  $P, Q$ . (Add. Prop. 24.)

Similarly it can be proved that circles can be described through  $Q, R$ , and through  $R, P$  to touch at each pair of points.

Therefore these three circles are identical (see Ex. 140, page 284),

that is, the circle  $PQR$  touches the given circles at  $P, Q, R$ .

Similarly the circle  $pqr$  touches the given circles at  $p, q, r$ .

Since there are four axes of similitude, and since two circles can be obtained by the foregoing construction from each axis of similitude, there are  $2 \times 4$  or 8 circles which can be described to touch three given circles.

It is readily seen that, if three circles touch a fourth circle, there are two distinct possible types of configuration of the three circles relatively to the circle which they touch :

- (1) the three circles may lie on the same side of the fourth circle,
- (2) two of the three circles may lie on one side and the third on the other side of the fourth circle.

Since any one of the three given circles may be the one which lies by itself on the one side of the fourth circle, the type (2) may be subdivided into three different groups.

It can be proved without much difficulty that of the eight circles which can be described to touch three given circles  $A, B, C$ , there are four pairs of circles, such that

$A, B, C$  lie on the same side of each circle of one pair ;

$A$  lies on one side, and  $B, C$  on the other, of each circle of a second pair ;

$B$  lies on one side, and  $C, A$  on the other, of each circle of a third pair ;

and  $C$  lies on one side, and  $A, B$  on the other, of each circle of a fourth pair.

Each of these four pairs of circles is obtained by the above construction from one of the axes of similitude of the given circles.

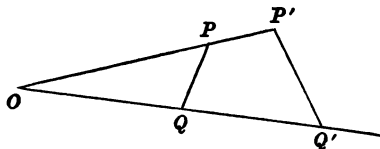
## INVERSION.

1. We will now proceed to give an account of a geometrical Method called Inversion, and we will do so without stating the theorems which we are about to establish in the formal way in which theorems have been stated heretofore. We will further depart from our former method by making use of the arithmetical method of representing geometrical magnitudes (see page 135), so that we shall not be debarred from using fractions to represent the ratios of geometrical quantities, and if necessary we shall use the signs ordinarily used in Algebra to signify addition, subtraction, and the other elementary operations.

2. DEFINITION. If  $O$  be a fixed point and  $P$  any other point, and if on the straight line  $OP$  (produced if necessary) we take a point  $P'$  such that

$$OP \cdot OP' = a^2,$$

where  $a$  is a constant, then each of the points  $P, P'$  is called the *inverse* of the other with respect to the circle whose centre is  $O$  and radius  $a$ .



The straight line  $OP$  is often called the *radius vector* of the point  $P$ .

The point  $O$  is called the *pole of inversion* and  $a$  the *radius of inversion*.

If the point  $P$  trace out a curve, the curve which is the locus of  $P'$  is called the *inverse of the curve* which is the locus of  $P$ .

8. If we consider the points  $P', P''$ , which are the inverses of  $P$  with respect to the same pole and different radii of inversion  $a, b$ , since  $OP \cdot OP' = a^2$  and  $OP \cdot OP'' = b^2$ ,

therefore  $OP' : OP'' = a^2 : b^2$ .

It follows that since the points  $P', P''$  lie on the same radius vector, and  $OP' : OP''$  is a constant ratio, the loci of  $P', P''$  are two similar curves of which  $O$  is a centre of similitude.

4. If  $P, P'$ , and  $Q, Q'$  be two pairs of inverse points, then since

$$OP \cdot OP' = a^2 \text{ and } OQ \cdot OQ' = a^2,$$

therefore  $OP \cdot OP' = OQ \cdot OQ'$ ;

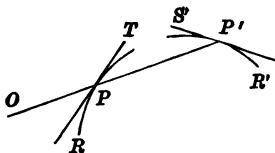
it follows that  $P, Q, Q', P'$  lie on a circle, and the angle  $OPQ$  is equal to the angle  $OQ'P'$ .

Again, if the line  $OQQ'$  approach nearer and nearer to the position of  $OPP'$ , the lines  $QP, Q'P'$  in the limit are the tangents to the curves which are the loci of  $P$  and  $P'$  at  $P$  and  $P'$  respectively.

(See page 217.)

We may state this result in the form :

*Two inverse curves at two inverse points cut the radius vector through the pole of inversion at the same angle on opposite sides.*



It follows that any two curves cut at the same angle as their inverse curves at the inverse point. (See Definition on page 266.)

5. From the definition of inversion it follows at once that a straight line through the pole inverts into itself.

Three points  $A, B, C$  in the same radius vector, whose distances  $OA, OB, OC$  are such that  $OA + OC = 2OB$ , invert into three points  $A', B', C'$ , whose distances  $OA', OB', OC'$  are such that

$$\frac{1}{OA'} + \frac{1}{OC'} = \frac{2}{OB'}.$$

Three magnitudes such as  $OA$ ,  $OB$ ,  $OC$  are said to be in **Arithmetical Progression**, and three magnitudes such as  $OA'$ ,  $OB'$ ,  $OC'$  are said to be in **Harmonical Progression**.

Because 
$$\frac{1}{OA'} + \frac{1}{OC'} = \frac{2}{OB'},$$
 therefore  $OB' \cdot OC' + OA' \cdot OB' = 2OA' \cdot OC'$ ;  
 whence  $OA' \cdot B'C' = OC' \cdot A'B'$ ,  
 and  $OA' : A'B' = OC' : C'B'$ ;  
 therefore  $OA'B'C'$  is a harmonic range.

Also three points  $A$ ,  $B$ ,  $C$  in the same radius vector, whose distances  $OA$ ,  $OB$ ,  $OC$  are such that  $OA \cdot OC = OB^2$ , invert into three points  $A'$ ,  $B'$ ,  $C'$ , whose distances are such that  $OA' \cdot OC' = OB'^2$ .

Three magnitudes such as  $OA$ ,  $OB$ ,  $OC$  are said to be in **Geometrical Progression**.

It may be noticed that three magnitudes  $a$ ,  $b$ ,  $c$  are in Arithmetical, Geometrical, or Harmonical Progression according as

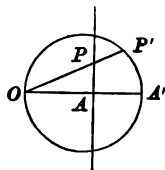
$$a - b : b - c = a : a,$$

$$a - b : b - c = a : b,$$

or

$$a - b : b - c = a : c \text{ respectively.}$$

6. Let  $P$  be a point on a fixed straight line; draw  $OA$  perpendicular to the line, and on  $OA$ , produced if necessary, take the point  $A'$  such that  $OA \cdot OA' = a^2$ .

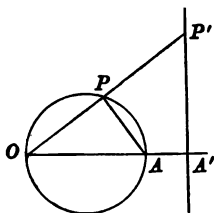


Let  $P'$  be the inverse of  $P$ , so that  $OP \cdot OP' = a^2$ : then  $OP \cdot OP' = OA \cdot OA'$ ; therefore  $P$ ,  $A$ ,  $A'$ ,  $P'$  lie on a circle, and the angle  $OP'A' =$  the angle  $OAP =$  a right angle; therefore the locus of  $P'$  is a circle whose diameter is  $OA'$ . Hence the theorem:

*A straight line which does not pass through the pole of inversion inverts into a circle which passes through the pole and touches there a parallel to the given line.*



7. Again let  $P$  be any point on a circle, of which  $OA$  is a diameter. Take in  $OA$ , produced if necessary, the point  $A'$  such that  $OA \cdot OA' = a^2$ .



Let  $P'$  be the inverse of  $P$ , so that  $OP \cdot OP' = a^2$ ;

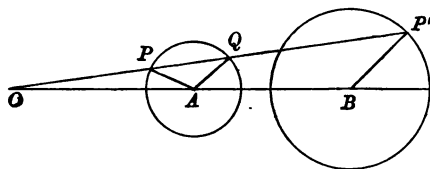
then  $OP \cdot OP' = OA \cdot OA'$ ;

therefore  $P, A, A', P'$ , lie on a circle,

and the angle  $P'A'O =$  the angle  $OPA =$  a right angle, i.e. the locus of  $P'$  is a straight line through  $A'$  the inverse of  $A$  drawn at right angles to  $OA'$ . Hence the theorem:

*A circle which passes through the pole of inversion inverts into a straight line parallel to the tangent at the pole.*

8. Again, let  $P$  be any point on a circle, which does not pass through the pole, and  $P'$  be the inverse point so that  $OP \cdot OP' = a^2$ : let  $OP$ , produced if necessary, cut the circle in  $Q$ ; find  $A$  the centre and draw  $QA$ , and let  $OA$ , produced if necessary, cut  $P'B$  drawn parallel to  $QA$  in  $B$ .



Because  $OP \cdot OQ = t^2$ , where  $t$  represents the length of the tangent from  $O$  to the given circle, and  $OP \cdot OP' = a^2$ , therefore  $OP' : OQ = a^2 : t^2 = OB : OA = BP' : AQ$ ; therefore the locus of  $P'$  is a circle such that  $B$  is its centre and  $O$  is one of the centres of similitude of it and the given circle.

Hence the theorem: *A circle, which does not pass through the pole of inversion, inverts into a circle which does not pass through the pole, and is such that the pole is a centre of similitude of the two circles.*

9. Again, if  $P, P'$  and  $Q, Q'$  be two pairs of inverse points,

$$OP \cdot OP' = OQ \cdot OQ' = a^2,$$

the triangles  $OPQ, OQ'P'$  are similar.

Therefore 
$$\frac{P'Q'}{PQ} = \frac{OQ'}{OP} = \frac{OQ' \cdot OQ}{OP \cdot OQ} = \frac{a^2}{OP \cdot OQ},$$

or 
$$P'Q' = a^2 \cdot \frac{PQ}{OP \cdot OQ}.$$

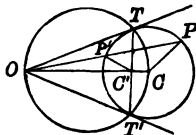
Thus the distance between two points in a figure is expressed in terms of the distance between their inverse points and the distances of the inverse points from the pole.

10. Now let us take the ordinary definition of a circle, i.e. the locus of a point  $P$  such that its distance  $PC$  from a fixed point  $C$  is constant.

Let  $P'$  be the inverse point of  $P$ ; take  $C'$  the inverse point of  $C$ .

If we please we may in this investigation choose the radius of inversion (see page 461) equal to the tangent drawn from  $O$  to the original circle; then  $P'$  lies on the same circle as  $P$ ,

and 
$$OC \cdot OC' = OT^2 = OP \cdot OP'.$$



Therefore the triangles  $OP'C', OCP$  are similar,

and 
$$OP' : P'C' = OC : CP.$$

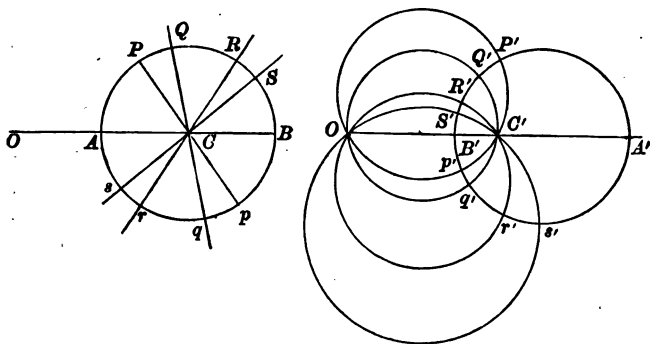
Hence the theorem (which has been already proved otherwise, Add. Prop. 5):

*The locus of a point the distances of which from two fixed points are in a constant ratio is a circle.*

It will be seen that the straight line  $TT'$ , which is the polar of  $O$ , and the circle on  $OC$  as diameter are inverses of each other.

11. Again, it is known from Book III. that all straight lines which cut a circle at right angles pass through the centre, and also that all straight lines through the centre cut the circle at right angles.

If therefore we invert a figure consisting of a series of straight lines cutting a series of concentric circles (centre  $C$ ) at right angles, we shall obtain a series of coaxial circles passing through  $O$  the pole of inversion and through  $C'$  the inverse point of  $C$ , and cutting each of a second series of circles at right angles. (Add. Prop. page 268.)



Hence the theorem :

*A system of concentric circles inverts into a system of coaxial circles.*

In the above diagram  $O$  is the pole of inversion, and the tangent from  $O$  to the circle  $PQR$  (or  $P'Q'R'$ ) is taken as the radius of inversion so that the circle inverts into itself. The points  $O, A, B$  in the left-hand figure are supposed to coincide with  $O, B', A'$  respectively in the right-hand figure : the figures are drawn apart merely for the sake of clearness.

Further, if we invert the last system with respect to any point, we shall get a system of exactly the same nature; viz. a system of circles passing through two fixed points and cutting each of another system of circles at right angles.

Hence the theorem :

*A system of coaxial circles inverts into another system of coaxial circles, and the limiting points into the limiting points.*

12. A circle can be inverted into itself with respect to any point as pole of inversion, if the tangent to the circle from the point be taken as the radius of inversion.

Any two circles can be inverted into themselves with respect to any point on their radical axis as the pole of inversion.

Any three circles can be inverted into themselves with respect to their radical centre as the pole of inversion.

Hence we conclude that when one circle is drawn to touch two given circles at  $P$  and  $Q$ , if  $O$  be any point on the radical axis of the given circles, the lines  $OP$ ,  $OQ$  will cut the circles again in two points  $P'$ ,  $Q'$ , such that another circle can be described to touch the given circles at  $P'$ ,  $Q'$ .

We also conclude that when one circle is drawn to touch three given circles at  $P$ ,  $Q$ ,  $R$ , if  $O$  be the radical centre, the lines  $OP$ ,  $OQ$ ,  $OR$  cut the circles again in three points  $P'$ ,  $Q'$ ,  $R'$  such that a circle described through them will touch the circles at  $P'$ ,  $Q'$ ,  $R'$ .

These two theorems have been proved before (page 455).

13. If two circles  $PQO$ ,  $pqO$  be described to touch two given circles  $A$ ,  $B$  at  $P$ ,  $p$ ;  $Q$ ,  $q$ , and to touch each other at  $O$ , then the centre of similitude  $F$ , which is the point of intersection of the straight lines  $PQ$ ,  $pq$ , must lie on the common tangent to the circles at  $O$ .

(Add. Prop. 29.)

Hence, if  $O$  be taken as the pole of inversion, the two circles  $OPQ$ ,  $Opq$  will invert into two parallel common tangents to the inverse circles (page 423), and therefore the given circles will invert into a pair of equal circles.

And since  $FO^2 = FP \cdot FQ = FM \cdot FN$ , if  $FMN$  be a common tangent to the given circles,

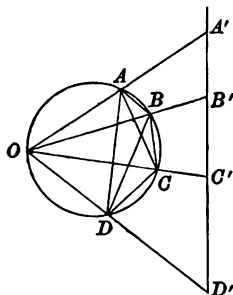
(Add. Prop. 26)

we see that the pole  $O$  may be chosen anywhere on the circle, whose centre is  $F$  and whose radius is a mean proportional between the tangents from the point  $F$  to the circles  $A$ ,  $B$ .

Hence with any point on two definite circles as pole of inversion we can invert two given circles into two equal circles.

It follows that with any one of certain definite points as pole of inversion, *three given circles can be inverted into three equal circles.*

14. If we take  $A', B', C'$  three points in order on a straight line, the relation  $A'B' + B'C' = A'C'$  exists between the segments.



If we invert with respect to any pole  $O$ , the three inverse points  $A, B, C$  will lie on a circle through  $O$ , and the chords will satisfy the relation

$$\frac{AB}{OA \cdot OB} + \frac{BC}{OB \cdot OC} = \frac{AC}{OA \cdot OC}, \quad (\text{see page 464}),$$

or

$$AB \cdot OC + BC \cdot OA = AC \cdot OB,$$

which is Ptolemy's Theorem.

(III. Prop. 37 B.)

Again if we take  $A', B', C', D'$ , four points in order on a straight line, the relation

$$A'B' \cdot C'D' + A'D' \cdot B'C' = A'C' \cdot B'D'$$

exists between the segments.

(Ex. 1, page 137.)

If we invert with respect to any pole  $O$ , the four inverse points  $A, B, C, D$  will lie on a circle through  $O$ , and the chords will satisfy the relation,

$$\frac{AB}{OA \cdot OB} \cdot \frac{CD}{OC \cdot OD} + \frac{AD}{OA \cdot OD} \cdot \frac{BC}{OB \cdot OC} = \frac{AC}{OA \cdot OC} \cdot \frac{BD}{OB \cdot OD},$$

or

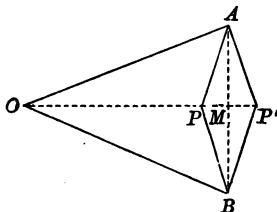
$$AB \cdot CD + AD \cdot BC = AC \cdot BD,$$

which also is a form of Ptolemy's Theorem.

## PEAUCELLIER'S CELL.

15. Before we leave the subject of inversion, it will be as well to explain the nature of a simple piece of mechanism, by the use of which the inverse of any given curve can be drawn.

The figure represents such an instrument;  $OA, OB$  are two equal rods and  $AP, PB, BP', P'A$ , four other equal rods, all six being freely hinged together at the points  $O, A, B, P, P'$ .



This instrument is generally called a Peaucellier's Cell\*.

Because  $APBP'$  is a rhombus,  
its diagonals bisect each other at right angles at  $M$ ; (Ex. 1, page 39)  
and because  $OAB$  is an isosceles triangle,  
and  $M$  is the middle point of  $AB$ ,  
 $OM$  is at right angles to  $AB$ .

Therefore  $OPP'$  is a straight line,

$$\begin{aligned}\text{and the rectangle } OP \cdot OP' &= OM^2 - MP^2 \text{ (II. Prop. 6)} \\ &= OA^2 - AP^2 \\ &= \text{a constant.}\end{aligned}$$

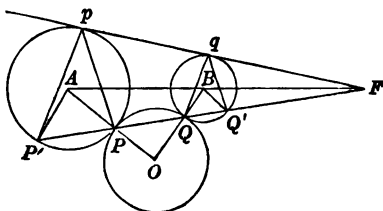
If therefore the point  $O$  be fixed, and  $P$  be made to trace out any curve,  $P'$  will trace out the inverse curve.

\* This mechanical invention is due to A. Peaucellier, Capitaine du Génie (à Nice), who proposed the design of such an instrument as a question for solution in the *Nouvelles Annales* 1864 (p. 414). His solution was published in the *Nouvelles Annales* 1873 (pp. 71—8).

We will now proceed to prove a theorem which establishes a relation between six of the common tangents of pairs of four circles which touch a fifth given circle, of a kind similar to that which Ptolemy's Theorem establishes between the chords joining the points of contact.

Let a circle whose centre is  $O$  touch two given circles whose centres are  $A, B$  at  $P, Q$ ; let  $PQ$ , produced if necessary, cut the circles  $A, B$  again in  $P', Q'$  and pass through their centre of similitude  $F$ ; and let  $pqF$  be one of their common tangents through  $F$ .

(Add. Prop. 24 note.)



Because  $F$  is a centre of similitude of the circles,

$pP'$  is parallel to  $qQ$ , and  $pP$  to  $qQ'$ ; (Add. Prop. 25)

therefore  $pq : P'Q = Fq : FQ$ ,

and  $pq : PQ' = Fq : FQ'$ ;

therefore  $pq^2 : P'Q \cdot PQ' = Fq^2 : FQ \cdot FQ'$ ;

and  $Fq^2 = FQ \cdot FQ'$ ;

therefore  $pq^2 = P'Q \cdot PQ'$ . (Ex. 1, page 453.)

Again because  $F$  is a centre of similitude,

$OPA$  is parallel to  $Q'B$ , and  $OQB$  to  $P'A$ ;

therefore  $OA : OP = PQ : PQ$ ,

and  $OB : OQ = PQ' : PQ$ ;

therefore  $OA \cdot OB : OP \cdot OQ = PQ' \cdot P'Q : PQ^2$ ,

or  $OA \cdot OB : OP^2 = pq^2 : PQ^2$ ;

therefore, if  $OA \cdot OB = OL^2$ ,

then  $OL : OP = pq : PQ$ . (V. Prop. 16.)

Similarly, if two other circles, whose centres are  $C, D$ , touch the same circle, centre  $O$ , at  $R, S$ , and  $rs$  be their common tangent,

$$OC \cdot OD : OP^2 = rs^2 : RS^2;$$

$$\text{therefore, if } OC \cdot OD = OM^2,$$

$$\text{then } OM : OP = rs : RS; \quad (\text{V. Prop. 16})$$

$$\text{therefore } OL \cdot OM : OP^2 = pq : rs = PQ \cdot RS.$$

If we denote the common tangent to the two circles which touch the fifth circle at  $P, Q$  by  $(PQ)$ , and so on for the other pairs of circles, we may write this proportion,

$$OL \cdot OM : OP^2 = (PQ) \cdot (RS) : PQ \cdot RS.$$

It can be proved in a similar manner that,

$$\text{if } OL'^2 = OA \cdot OC \text{ and } OM'^2 = OB \cdot OD,$$

$$\text{then } OL' \cdot OM' : OP^2 = (PR) \cdot (QS) : PR \cdot QS.$$

$$\text{Now } OL^2 : OL'^2 = OA \cdot OB : OA \cdot OC = OB : OC,$$

$$\text{and } OM'^2 : OM^2 = OB \cdot OD : OC \cdot OD = OB : OC;$$

$$\text{therefore } OL^2 : OL'^2 = OM'^2 : OM^2,$$

$$\text{and } OL : OL' = OM' : OM; \quad (\text{V. Prop. 16})$$

$$\text{therefore } OL \cdot OM = OL' \cdot OM'. \quad (\text{Prop. 16.})$$

Hence we have

$$(PQ) \cdot (RS) : PQ \cdot RS = (PR) \cdot (QS) : PR \cdot QS$$

$$\text{and similarly } = (PS) \cdot (QR) : PS \cdot QR.$$

And because, if  $PQRS$  be a convex quadrilateral,

$$PR \cdot QS = PQ \cdot RS + PS \cdot QR \quad (\text{Ptolemy's theorem}),$$

$$\text{therefore } (PR) \cdot (QS) = (PQ) \cdot (RS) + (PS) \cdot (QR).$$

This theorem is generally known as **Casey's Theorem**\*.

It may be observed that the common tangent of each pair of circles, which appears in the equation, is that tangent which passes through the same centre of similitude of the circles as the chord joining the points of contact of the circles with the fifth circle.

\* This theorem was discovered by John Casey (born at Kilbenny, County Cork, 1820, died at Dublin 1890).



It is readily seen that, if four circles touch a fifth circle, there are three distinct possible types of configuration of the four circles relatively to the circle which they touch ;

- (1) the four circles may lie on the same side of the fifth circle,
- (2) three of the four circles may lie on one side and the fourth circle on the other side,
- (3) two of the four circles may lie on one side and the other two on the other side.

The converse of Casey's Theorem may be stated in the following manner: *if an equation of the form of the equation in Casey's Theorem exist between the common tangents of four circles taken in pairs, the common tangents being chosen in accordance with one of the three possible types of configuration, then the four circles touch a fifth circle.*

The truth of this converse theorem which is often assumed without any attempt at proof can be proved, but the proof of it is thought to be beyond the scope of this work.

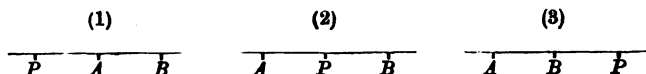
## CONTINUITY.

Let us consider a variable point  $P$  on a given straight line, on which  $A, B$  are two fixed points. It is seen at once that,

(1) if  $P$  be outside  $AB$  beyond  $A$ , then the excess of  $PB$  over  $PA$  is equal to  $AB$ ;

(2) if  $P$  be in  $AB$ , then the sum of  $AP$  and  $PB$  is equal to  $AB$ ,  
and

(3) if  $P$  be outside  $AB$  beyond  $B$ , then the excess of  $AP$  over  $BP$  is equal to  $AB$ .



We can write these results in the forms

$$(1) \quad PA + AB = PB,$$

$$(2) \quad AB = AP + PB,$$

$$(3) \quad AB + BP = AP.$$

Here we observe that, while  $P$  changes from one side of  $A$  to the other, the distance  $PA$ , which vanishes when  $P$  coincides with  $A$ , changes sides in the equation, which otherwise remains unchanged; and again that while  $P$  changes from one side of  $B$  to the other, the distance  $PB$  similarly changes sides in the equation.

A geometrical theorem consists, in many cases, of a proof that a certain equation exists between a number of geometrical magnitudes, such equation remaining unchanged in form for variations in the geometrical magnitudes involved, consistent with the conditions to which they are subject.

It is found in many of such theorems, as in the illustration which we have just given, that, if subject to continuous variation of some chosen geometrical magnitude some other magnitude continuously diminish and vanish, then in the equation which applies to the configuration determined by the next succeeding values of the chosen variable magnitude, the magnitude which has vanished appears on the opposite side of the equation. This fact is due to the absence of any sudden changes in the magnitudes under consideration. The general law that no sudden change occurs is often spoken of as THE PRINCIPLE OF CONTINUITY.

Let us consider a variable point  $P$  on a given straight line, on which  $A$  is a fixed point; and let us consider any equation between variable geometrical magnitudes, one of which is  $PA$  the distance between  $P$  and  $A$ . The principle of continuity leads us to expect that, if  $P$  in the variation of its position pass from one side to the other of  $A$ , the sign of  $PA$  in the equation will change. In other words, we may consider the equation to remain unchanged in form, if we resolve to represent by the expression  $PA$  not only the distance between  $P$  and  $A$ , but also the fact that the distance is measured from  $P$  towards  $A$ . This result is at once obtained by resolving that  $-PA$  shall represent a distance equal to  $PA$  and measured in the opposite direction; in other words, that  $AP = -PA$ .

Let us return to the consideration of a variable point  $P$  on a given straight line, on which  $A, B$  are two fixed points. It appears that the equation which exists between the distances between the points takes different forms according as  $P$  is (1) in  $BA$  produced, (2) in  $AB$ , or (3) in  $AB$  produced.

If we allow the use of the *minus* sign, we may write these equations,

$$(1) \quad AB + PA - PB = 0,$$

$$(2) \quad AB - AP - PB = 0,$$

$$(3) \quad AB - AP + BP = 0,$$

where each symbol such as  $AB$  represents merely the length of a line measured in the same direction as  $AB$ .

It is at once seen that, if we adopt the convention that

$$PQ = -QP,$$

all these equations are the same; each may be written

$$AB = AP + PB,$$

or

$$AB + BP + PA = 0.$$

The first form expresses that the operation of passing from  $A$  to  $B$  is the same as passing from  $A$  to  $P$  and then from  $P$  to  $B$ ;  
 the second form expresses that the aggregate result of the operations of passing from  $A$  to  $B$ , and then from  $B$  to  $P$  and then from  $P$  to  $A$  is to arrive at the point  $A$  of starting;  
 both of which facts are true for *all* combinations of three points  $A, B, P$  on a straight line.

The results of the theorems contained in Propositions 5 and 6 of Book II. become the same, if we take into account the fact that the distance  $BD$  is measured in opposite directions in the two figures: and similarly the results of Propositions 12 and 13 of Book II. become the same, if we take into account the sign of  $CD$ .

As a further illustration of the Principle of Continuity we will take Ptolemy's Theorem.

Let us consider a variable point  $P$  on a circle, on which  $A, B, C$  are three fixed points. It is proved in III. Prop. 37 B, that

$$(1) \text{ if } P \text{ be in the arc } AB, \\ AB \cdot PC = BC \cdot PA + CA \cdot PB;$$

$$(2) \text{ if } P \text{ be in the arc } BC, \\ BC \cdot PA = CA \cdot PB + AB \cdot PC;$$

and

$$(3) \text{ if } P \text{ be in the arc } CA, \\ CA \cdot PB = AB \cdot PC + BC \cdot PA.$$

These equations may be written

$$(1) \quad AB \cdot PC - BC \cdot PA - CA \cdot PB = 0,$$

$$(2) \quad AB \cdot PC - BC \cdot PA + CA \cdot PB = 0,$$

$$(3) \quad AB \cdot PC + BC \cdot PA - CA \cdot PB = 0.$$

Hence, while  $P$  passes along the arc from one side of  $B$  to the other, the sign of  $PB$ , which vanishes when  $P$  coincides with  $B$ , changes sign in the equation, which otherwise remains unchanged, and so on for passage through  $C$  or  $A$ .

# PORISMATIC PROBLEMS.

In some cases, when a Geometrical problem is submitted for solution, it is found that some relation between the geometrical magnitudes or figures which are given is necessary in order that a solution may be possible, and that, if one solution be possible, and therefore the relation exist, the number of possible solutions is infinite. The solution is then said to be *indeterminate*.

Such a problem is called a PORISM.

We might take as an illustration of such a problem the problem, to construct a triangle which shall be inscribed in one and described about another of two given concentric circles. It is easily seen that, if one such triangle exist, the radius of the first circle must be equal to twice the radius of the second, and that then every equilateral triangle which is inscribed in the first circle is also described about the second circle.

Again we might take the problem, to construct a triangle so that each vertex shall lie on one of three given concentric circles, and that each side shall touch a fourth given concentric circle. It is easily proved by means of the method of rotation (page 186) that, if such a triangle be possible, an infinite number of such triangles are possible.

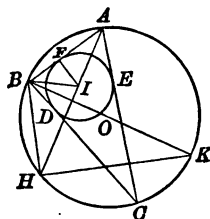
We will now proceed to find the relation which must exist between two circles which are not concentric, in order that it may be possible to construct a triangle which shall be inscribed in one and described about the other, and we shall prove that, if it be possible to construct one such triangle, it is possible to construct an infinite number.

*The square on the straight line joining the centres of the circumscribed circle and the inscribed circle of a triangle is less than the square on the radius of the circumscribed circle by twice the rectangle contained by the radii of the two circles.*

Let  $ABC$  be the circumscribed circle of the triangle  $ABC$  and  $DEF$  the inscribed circle touching  $AB$  at  $F$ .

Find  $O, I$  the centres of the circles  $ABC, DEF$ . Draw  $AI, IB, BO, IF$  and let  $AI, BO$  produced meet the circle  $ABC$  at  $H, K$ .

Draw  $BH, HK$ .



Because the angle  $FAI$  (or  $BAH$ ) is equal to the angle  $HKB$ ,  
 and the angle  $AFI$  is equal to the angle  $KHB$ ,  
 the triangles  $FAI, HKB$  are equiangular to one another;  
 therefore the rectangle  $AI, BH$  is equal to the rectangle  $KB, IF$ .  
 (III. Prop. 37 A.)

Because the angle  $BIH$  is equal to the sum of the angles  $BAI, ABI$ ,  
 which is equal to the sum of the angles  $HBC, CBI$ ,  
 that is, to the angle  $IBH$ ;  
 therefore  $BH$  is equal to  $IH$ ;  
 therefore the rectangle  $AI, BH$  is equal to the rectangle  $AI, IH$ ,  
 which is equal to the difference of the squares on  $OB, OI$ ;  
 (III. Prop. 35)  
 therefore the difference of the squares on  $OB, OI$  is equal to the rectangle  $KB, IF$ ,  
 that is, to twice the rectangle contained by the radii.

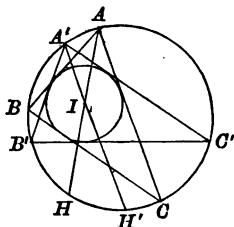
If two circles be such that one triangle can be constructed that is inscribed in one circle and circumscribed about the other circle, an infinite number of such triangles can be constructed.

Let  $ABC$  be a triangle: and let its circumscribed and inscribed circles be described, and let  $I$  be the centre of the inscribed circle.

Take any other point  $A'$  on the circumscribed circle  $ABC$ ; draw  $AI$ ,  $A'I$  and produce them to cut the circle  $ABC$  at  $H$ ,  $H'$ .

With centre  $H'$  and radius  $H'I$  draw a circle cutting the circle  $ABC$  in  $B'$ ,  $C'$ .

Draw  $A'B'$ ,  $B'C'$ ,  $C'A'$ .



Because  $H'B'$  is equal to  $H'I$ ,

the angle  $H'IB'$  is equal to the angle  $H'B'I$ ;

therefore the sum of the angles  $IB'A'$ ,  $IA'B'$  is equal to the sum of the angles  $H'B'C'$ ,  $C'B'I$ . (I. Prop. 32.)

And because the chords  $H'B'$ ,  $H'C'$  are equal,

the angle  $IA'B'$  is equal to the angle  $H'B'C'$ ;

therefore the angles  $IB'A'$ ,  $C'B'I$  are equal,

that is,  $B'I$  is the bisector of the angle  $A'B'C'$ .

And  $A'I$  is the bisector of the angle  $B'A'C'$ ;

therefore  $I$  is the centre of the inscribed circle of the triangle  $A'B'C'$ .

Because  $AIH$ ,  $A'IH'$  are two chords of the circle  $ABC$ ,

the rectangle  $AI$ ,  $IH$  is equal to the rectangle  $A'I$ ,  $IH'$ ;

and the rectangle  $AI$ ,  $IH$  is equal to twice the rectangle contained by the radius of the circle  $ABC$  and the radius of the inscribed circle of the triangle  $ABC$ , (page 476)

and the rectangle  $A'I$ ,  $IH'$  is equal to twice the rectangle contained by the radius of the circle  $ABC$  and the radius of the inscribed circle of the triangle  $A'B'C'$ . (page 476.)

Therefore the radii of the inscribed circles of the triangles  $ABC$ ,  $A'B'C'$  are equal:

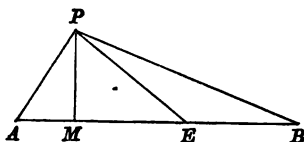
and the circles have a common centre  $I$ .

Therefore the triangle  $A'B'C'$  has the same circumscribed circle and the same inscribed circle as the triangle  $ABC$ .

We now proceed to prove a theorem which might fairly have been included at an earlier stage.

Let  $PAB$  be any triangle and  $E$  be a point in  $AB$ , such that  $mAE = nEB$ . Here  $E$  is the centroid of weights  $m$  and  $n$  at  $A$  and  $B$ . (See page 425.)

Draw  $PM$  perpendicular to  $AB$  and draw  $PE$ .



Because  $PA^2 = PE^2 + EA^2 - 2ME \cdot AE$ ; (II. Prop. 13)  
 and  $PB^2 = PE^2 + EB^2 + 2ME \cdot EB$ , (II. Prop. 12)  
 and because  $mAE = nEB$ ,  
 and therefore  $mME \cdot AE = nME \cdot EB$ ;  
 therefore  $mPA^2 + nPB^2 = (m+n)PE^2 + mEA^2 + nEB^2$ .

Hence the theorem:

*The sum of any multiples of the squares on the distances of any point from two given points is equal to the sum of the same multiples of the squares on the distances of the given points from the centroid of weights at the given points proportional to those multiples, together with the sum of the multiples of the square on the distance of the point from the centroid.*

Because  $mAE = nEB$ ,  
 therefore  $mAE^2 = nAE \cdot EB$ ,  
 and  $mAE^2 + nEB^2 = nAE \cdot EB + nEB^2$   
 $= nAB \cdot EB$ .

We may therefore write the result of this theorem in the form

$$mPA^2 + nPB^2 = (m+n)PE^2 + nBE \cdot BA$$

or  $nPB^2 - nBE \cdot BA = (m+n)PE^2 - mPA^2$ .

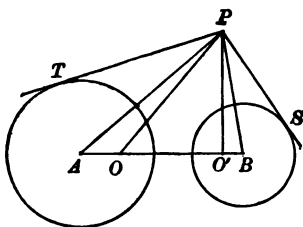


Next, let us assume  $P$  a point such that the tangents  $PT$ ,  $PS$  drawn from it to two given circles, whose centres are  $A$ ,  $B$ , satisfy the equation  $mPT = nPS$ .

Take two points  $O$ ,  $O'$  in  $AB$ , such that

$$AO \cdot AO' = AT^2 \text{ and } BO \cdot BO' = BS^2.$$

(Add. Prop. page 268.)



Because

$$mPT = nPS,$$

$$m^2 PT^2 = n^2 PS^2;$$

therefore

$$m^2 (PA^2 - AT^2) = n^2 (PB^2 - BS^2),$$

or

$$m^2 (PA^2 - AO \cdot AO') = n^2 (PB^2 - BO \cdot BO').$$

Now by the last theorem, if  $qAO = pOO'$  and  $rOO' = qO'B$ ,

$$q(PB^2 - BO \cdot BO') = (r+q)PO'^2 - rPO^2,$$

and

$$q(PA^2 - AO \cdot AO') = (p+q)PO^2 - pPO'^2.$$

Therefore

$$n^2(q+r)PO'^2 - n^2rPO^2 = m^2(p+q)PO^2 - m^2pPO'^2,$$

or

$$\{m^2(p+q) + n^2r\}PO^2 = \{n^2(q+r) + m^2p\}PO'^2.$$

Therefore  $PO$  is to  $PO'$  in a constant ratio, and therefore the locus of  $P$  is one of the system of coaxial circles, of which  $O$ ,  $O'$  are the limiting points. (See Note on page 429.)

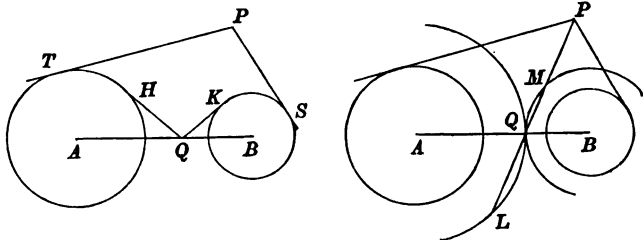
In consequence of the very algebraical character of the proof which has just been given, we will give another proof of the same theorem depending in a great measure on the theory of similitude.

Let us again assume  $P$  is a point such that the tangents  $PT$ ,  $PS$  drawn to two given circles, whose centres are  $A$ ,  $B$ , satisfy the equation

$$mPT = nPS.$$

There must be some point  $Q$  in  $AB$ , such that, if  $QH$ ,  $QK$  be the tangents to the circles,

$$\begin{aligned} & mQH = nQK; \\ \text{therefore} & m^2 PT^2 = n^2 PS^2, \\ \text{and} & m^2 QH^2 = n^2 QK^2; \\ \text{therefore} & m^2 (PT^2 - QH^2) = n^2 (PS^2 - QK^2), \\ \text{or} & m^2 (PA^2 - AQ^2) = n^2 (PB^2 - BQ^2). \end{aligned}$$



Now draw two circles, with centres  $A$  and  $B$ , and radii  $AQ$ ,  $BQ$ , and let  $PQ$  cut these circles in  $L$ ,  $M$  respectively; then

$$\begin{aligned} & PA^2 - AQ^2 = PQ \cdot PL, \\ \text{and} & PB^2 - BQ^2 = PQ \cdot PM; \quad (\text{III. Prop. 36}) \\ \text{therefore} & m^2 PQ \cdot PL = n^2 PQ \cdot PM, \\ \text{or} & m^2 PL = n^2 PM. \end{aligned}$$

$$\begin{aligned} \text{Therefore} & m^2 (PQ + QL) = n^2 (PQ - QM), \\ \text{or} & (n^2 - m^2) PQ = m^2 QL + n^2 QM. \\ \text{Therefore} & (n^2 - m^2) PQ : QM = m^2 QL + n^2 QM : QM \\ & = m^2 QA + n^2 QB : QB, \\ \text{since} & QL : QM = QA : QB. \end{aligned}$$

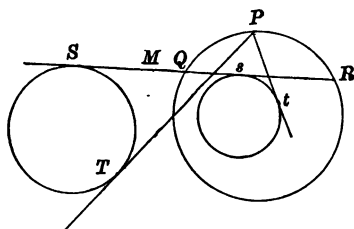
From this it follows that the ratio  $QP : QM$  is constant; therefore the locus of  $P$  is a circle which passes through  $Q$ , and has its centre in the line  $AB$ . (See page 450.)

Hence the theorem: *the locus of a point, such that the tangents drawn from it to two given circles are in a constant ratio, is a circle.*

Next, let  $P$  be a point on the locus such that  $PT$ ,  $Pt$ , the tangents drawn to two given circles, are in a given ratio.

Draw  $Ss$  a common tangent to the two given circles.

Let  $Ss$  cut the radical axis of the given circles in  $M$ , and the circle which is the locus of  $P$  in  $Q$  and  $R$ .



Because both  $Q$  and  $R$  are points on the locus of  $P$ ,

$QS$  is to  $Qs$  as  $RS$  to  $Rs$ ;

therefore  $SQsR$  is a harmonic range.

And because  $M$  is the middle point of  $Ss$ ,

$$MS^2 = Ms^2 = MQ \cdot MR,$$

and  $MQ \cdot MR$  is equal to the square on the tangent from  $M$  to the locus of  $P$ .

It follows therefore that  $M$  is a point on the radical axis of each of the pairs of circles; and since their centres are collinear, the circles have a common radical axis.

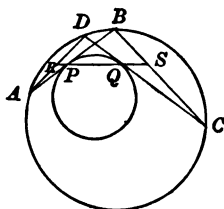
Hence the theorem:

*The circle which is the locus of a point, such that the tangents drawn from it to two given circles are in a constant ratio, belongs to the same coaxial system as the given circles.*

We will now prove the theorem :

*If two opposite sides of a quadrilateral, which is inscribed in a circle, touch another circle, the other sides of the quadrilateral touch a third circle coaxial with the other two.*

Let  $ABCD$  be a quadrilateral inscribed in a given circle ;  
and let  $AB, CD$  touch another given circle at  $P, Q$ .  
Draw  $PQ$  and let it be produced to meet  $AD, BC$  in  $R, S$ .



Because the angles  $BPS, DQR$  are equal  
and the angles  $PBS, QDR$  are equal, (III. Prop. 21)  
therefore the triangles  $BPS, DQR$  are equiangular to one another ;  
therefore  $BP$  is to  $BS$  as  $DQ$  to  $DR$ . (Prop. 4.)

Similarly it can be proved that the triangles  $APR, CQS$  are equiangular to one another and that  $AP$  is to  $AR$  as  $CQ$  to  $CS$ .

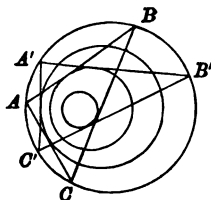
Again, because the angles  $BSP, DRQ$  are equal,  
a circle can be described to touch  $BC, DA$  at  $S, R$ .

Next, because in the two triangles  $ARP, BSP$ ,  
the angles  $APR, BPS$  are equal,  
and the angles  $ARP, BSP$  are supplementary ;  
therefore  $AP$  is to  $AR$  as  $BP$  to  $BS$ . (Prop. 5 A.)

Therefore the ratios of the tangents drawn from the four points  $A, B, C, D$  to the two circles  $PQ, RS$  are equal.

Therefore the four points  $A, B, C, D$  all lie on a circle coaxial with the two circles  $PQ, RS$  ; (page 481)  
that is, the circle  $ABCD$  is coaxial with the circles  $PQ, RS$ ,  
or, in other words, the circle  $RS$  is coaxial with the circles  $ABCD$  and  $PQ$ .

Next, let  $ABC$ ,  $A'B'C'$  be two triangles inscribed in a circle, such that  $AB$ ,  $A'B'$  touch a second circle and  $BC$ ,  $B'C'$  a third circle belonging to the same coaxial system.



Because  $ABB'A'$  is a quadrilateral inscribed in the circle  $ABC$ ,  
and  $AB$ ,  $B'A'$  touch another circle,  
therefore  $AA'$ ,  $BB'$  touch a circle of the same coaxial system ;  
and because  $BCC'B'$  is a quadrilateral inscribed in the circle  $ABC$ ,  
and  $BC$ ,  $C'B'$  touch another circle of the system ;  
therefore  $BB'$ ,  $CC'$  touch a circle of the system.  
Therefore  $AA'$ ,  $BB'$ ,  $CC'$  touch the same circle of the system.  
And because  $AA'C'C$  is a quadrilateral inscribed in the circle  $ABC$ ,  
and  $AA'$ ,  $C'C$  touch another circle of the system,  
therefore  $AC$ ,  $A'C'$  touch a circle of the system ;  
that is,  $A'C'$  always touches that circle of the system which  $AC$  touches,  
or, in other words,  $A'C'$  touches a definite circle of the system.

Hence the theorem :

*If two sides of a triangle inscribed in a given circle touch given circles of the same coaxial system as the first circle, the third side touches a fourth fixed circle of the system.\**

**COROLLARY.** *If all the sides but one of a polygon inscribed in a given circle touch given circles of the same coaxial system as the first circle, the remaining side touches another fixed circle of the system.\**

\* These theorems are due to Jean Victor Poncelet (born at Metz 1788, died at Paris 1867).

## MISCELLANEOUS EXERCISES.

1. Two triangles  $ABC$ ,  $BCD$  have the side  $BC$  common, the angles at  $B$  equal, and the angles  $ACB$ ,  $BDC$  right angles. Shew that the triangle  $ABC$  is to the triangle  $BCD$  as  $AB$  to  $BD$ .

2. The rectangle contained by two straight lines is a mean proportional between the squares described upon them.

3. Any polygons whatsoever described about a circle are to one another as their perimeters.

4. The sum of the perpendiculars drawn from any point within an equilateral triangle on the three sides is invariable.

5. In a parallelogram  $E, F, G, H$  are the middle points of the sides  $AB, BC, CD, DA$ ; if  $AF, AG, CE, CH$  be drawn, the parallelogram formed by them is one-third of the parallelogram  $ABCD$ .

6.  $ABC$  is a triangle,  $D$  any point in  $AB$  produced;  $E$  a point in  $BC$ , such that  $CE$  is to  $EB$  as  $AD$  to  $BD$ . Prove that  $DE$  produced bisects  $AC$ .

7.  $ABC, ABD$  are triangles on the same base, and  $CD$  meets the base in  $E$ ; then  $CE$  is to  $DE$  as the triangle  $ABC$  to the triangle  $ABD$ .

8. Triangles of unequal altitudes are to each other in the ratio compounded of the ratios of their altitudes and their bases.

9. If triangles  $ABC, AEF$  have a common angle  $A$ , the triangle  $ABC$  is to the triangle  $AEF$  as the rectangle  $AB, AC$  to the rectangle  $AE, AF$ .

10.  $O$  is the centre of the circle inscribed in a triangle  $ABC$ , and  $BO, CO$  meet the opposite sides in  $D, E$  respectively. Prove that the triangles  $BOE, COD$  are to one another in the ratio of the rectangles  $AE, AB; AD, AC$ .

11. If in the sides of a triangle  $BC, CA, AB$ , points  $D, E, F$  be taken such that  $BD$  is twice  $DC$ ,  $CE$  twice  $EA$ , and  $AF$  twice  $FB$ , and  $AD, BE, CF$  intersect in pairs in  $P, Q, R$ , then the areas of the triangles  $PQR, ABC$  are in the ratio of 1 to 7.

12. A point and a straight line being given, to draw a line parallel to the given line such that all the lines drawn through the point may be cut by the parallels in a given ratio.

13. If from a point  $O$  in the base  $BC$  of a triangle  $OM$  and  $ON$  be drawn parallel to the sides  $AB$  and  $AC$  respectively, then the area of the triangle  $AMN$  is a mean proportional between the areas of  $BNO$  and  $CMO$ .

14. If  $P, Q$  be two points within a parallelogram  $ABCD$ , and if  $PA, QB$  meet in  $R$ , and  $PD, QC$  meet in  $S$ , and if  $PQ$  be parallel to  $AB$ , then  $RS$  is parallel to  $AD$ .

15. A point  $P$  is taken on the bisector of the angle  $BAC$  of the triangle  $ABC$  between  $A$  and the base; prove that, if  $AC$  be greater than  $AB$ , the ratio  $PC$  to  $PB$  is greater than the ratio  $AC$  to  $CB$ .

16. On a circle of which  $AB$  is a diameter take any point  $P$ . Draw  $PC$ ,  $PD$  on opposite sides of  $AP$ , and equally inclined to it, meeting  $AB$  at  $C$  and  $D$ : prove that  $AC$  is to  $BC$  as  $AD$  is to  $BD$ .

17. Apply VI. 3 to solve the problem of the trisection of a finite straight line.

18.  $AB$  is a diameter of a circle,  $CD$  is a chord at right angles to it, and  $E$  is any point in  $CD$ ;  $AE$  and  $BE$  are drawn and produced to cut the circle at  $F$  and  $G$ : shew that the quadrilateral  $CFDG$  has any two of its adjacent sides in the same ratio as the remaining two.

19.  $ABCD$  is a quadrilateral having the opposite sides  $AD$ ,  $BC$  parallel;  $E$  is a point in  $AB$ , and the straight lines  $EC$ ,  $ED$  are drawn;  $AF$  is drawn parallel to  $EC$  meeting  $CD$  in  $F$ ; shew that  $BF$  is parallel to  $ED$ .

20. If a straight line  $AD$  be drawn bisecting the angle  $BAC$  of the triangle  $BAC$ , and meeting  $BC$  in  $D$ , and  $FDC$  be drawn perpendicular to  $AD$ , to meet  $AB$  and  $AC$  produced if necessary, in  $F$  and  $E$  respectively, and  $EG$  be drawn parallel to  $BC$ , meeting  $AB$  in  $G$ , then  $BG$  is equal to  $BF$ .

21. The side  $AB$  of the triangle  $ABC$  is produced to  $G$ , and the angle  $CBG$  bisected by  $BE$  meeting  $AC$  in  $E$ : the angle  $EBG$  is bisected by  $BL$  and the exterior angle of the triangle  $EBC$  got by producing  $EB$  is bisected by  $BF$ . Shew that, if  $BL$  be parallel to  $AC$  and  $BF$  meet  $AC$  in  $F$ ,  $CE$  is a mean proportional between  $CA$  and  $CF$ .

22. Each acute angle of a right-angled triangle and its corresponding exterior angle are bisected by straight lines meeting the opposite sides; prove that the rectangle contained by the portions of those sides intercepted between the bisecting lines is four times the square on the hypotenuse.

23.  $A$  and  $B$  are fixed points, and  $AP$ ,  $BQ$  are parallel chords of a variable circle, which passes through the fixed points. If the ratio of  $AP$  to  $BQ$  be constant, then the loci of  $P$  and  $Q$  are each a pair of circles, and the sum of the radii of two of these circles, and the difference of the radii of the other two, are independent of the magnitude of the ratio.

24. If two chords  $AB$ ,  $AC$ , drawn from a point  $A$  in the circumference of the circle  $ABC$ , be produced to meet the tangent at the other extremity of the diameter through  $A$  in  $D$ ,  $E$  respectively, then the triangle  $AED$  is similar to the triangle  $ABC$ .

25.  $AB$  is the diameter of a circle,  $E$  the middle point of the radius  $OB$ ; on  $AE$ ,  $EB$  as diameters circles are described;  $PQL$  is a common tangent meeting the circles at  $P$  and  $Q$ , and  $AB$  produced at  $L$ : shew that  $BL$  is equal to the radius of the smaller circle.

26. From  $B$  the right angle of a right-angled triangle  $ABC$ ,  $Bp$  is let fall perpendicular to  $AC$ ; from  $p$ ,  $pq$  is let fall perpendicular to  $BA$ ; from  $q$ ,  $qr$  is let fall perpendicular to  $AC$ , and so on; prove that

$$Bp + pq + \&c. : AB = AC : BC.$$

27. An isosceles triangle is described on a side of a square and the vertex joined with the opposite angles: the middle segment of the side has to either of the outside segments double of the ratio of the altitude of the triangle to its base.

28. A straight line  $DE$  is drawn parallel to the side  $BC$  of a triangle  $ABC$ .  $Q$  is a point in  $BC$  such that the rectangle  $BC, CQ$  is equal to the square on  $DE$ , and  $CR$  is taken equal to  $DE$  in  $BC$  produced. Prove that  $AR$  is parallel to  $DQ$ .

29.  $ABC, DEF$  are triangles, having the angle  $A$  equal to the angle  $D$ ; and  $AB$  is equal to  $DF$ : shew that the areas of the triangles are as  $AC$  to  $DE$ .

30.  $CA, CB$  are diameters of two circles which touch each other externally at  $C$ ; a chord  $AD$  of the former circle, when produced, touches the latter at  $E$ , while a chord  $BF$  of the latter, when produced, touches the former at  $G$ : shew that the rectangle contained by  $AD$  and  $BF$  is four times that contained by  $DE$  and  $FG$ .

31. If  $AA' B'B, BB' C'C, CC' A'A$  be three circles, and the straight lines  $AA', BB', CC'$  cut the circle  $A'B'C'$  again in  $\alpha, \beta, \gamma$  respectively, the triangle  $\alpha\beta\gamma$  will be similar to the triangle  $ABC$ .

32. On the two sides of a right-angled triangle squares are described: shew that the straight lines joining the acute angles of the triangle and the opposite angles of the squares cut off equal segments from the sides, and that each of these equal segments is a mean proportional between the remaining segments.

33.  $ABA'B'$  is a rectangle inscribed in a circle and  $AB$  is twice  $A'B$ ;  $AC$  is a chord equal to  $AB$  and meeting  $B'A'$  in  $F$  and  $BA'$  in  $E$ ; prove that  $AF : AE = CF : CA$ .

34. If  $BD, CD$  are perpendicular to the sides  $AB, AC$  of a triangle  $ABC$  and  $CE$  is drawn perpendicular to  $AD$ , meeting  $AB$  in  $E$ , then the triangles  $ABC, ACE$  are similar.

35. Describe a circle touching the side  $BC$  of the triangle  $ABC$  and the other two sides produced; and shew that the distance between the points of contact of  $BC$  with this circle and the inscribed circle is equal to the difference between  $AB$  and  $AC$ .

36. A straight line  $AB$  is divided into any two parts at  $C$ , and on the whole straight line and on the two parts of it equilateral triangles  $ADB, ACE, BCF$  are described, the two latter being on the same side of the straight line, and the former on the opposite side;  $G, H, K$  are the centres of the circles inscribed in these triangles: shew that the angles  $AGH, BGK$  are respectively equal to the angles  $ADC, BDC$ , and that  $GHK$  is an equilateral triangle.



37. Two circles, centres  $A$  and  $B$ , touch one another at  $C$ . A straight line is drawn cutting one circle in  $P$  and  $Q$  and the other circle in  $R$  and  $S$ . Prove that the ratio of the rectangle  $PR, PS$  to the square on  $CP$  is constant.

38.  $AB$  is the diameter of a circle,  $E$  the middle point of the radius  $OB$ ; on  $AE, EB$  as diameters circles are described.  $PQL$  is a common tangent, meeting the circles in  $P$  and  $Q$ , and  $AB$  produced in  $L$ . Shew that  $BL$  equals the radius of the lesser circle.

39. If through the vertex, and the extremities of the base of a triangle, two circles be described, intersecting one another in the base, or the base produced, their diameters are proportional to the sides of the triangle.

40.  $D$  is the middle point of the base  $BC$  of an isosceles triangle,  $CF$  perpendicular to  $AB$ ,  $DE$  perpendicular to  $CF$ ,  $EG$  parallel to the base meets  $AD$  in  $G$ ; prove that  $EG$  is to  $GA$  in the triplicate ratio of  $BD$  to  $DA$ .

41. Two straight lines and a point between them are given in position; draw two straight lines from the given point to terminate in the given straight lines, so that they shall contain a given angle and have a given ratio.

42. Four lines,  $AB, CD, EF, GH$ , drawn in any directions, intersect in the same point  $P$ ; then if from any point  $m$  in one of these lines, another be drawn parallel to the next in order, cutting the remaining two in  $p$  and  $q$ ; the ratio  $mp : pq$  is the same in whichever line the point  $m$  is taken.

43. If  $P, Q$  be the points of intersection of a variable circle drawn through two given points  $A, B$  with a fixed circle, prove that the ratio  $AP \cdot AQ : BP \cdot BQ$  is constant.

44. A quadrilateral is divided into four triangles by its diagonals; shew that if two of these triangles are equal, the remaining two are either equal or similar.

45.  $ABCD$  is a quadrilateral inscribed in a circle,  $E, F, G$  are the points of intersection of  $AB$  and  $CD$ ,  $AC$  and  $BD$ ,  $AD$  and  $BC$  respectively.  $K$  is the foot of the perpendicular let fall from  $F$  on  $EG$ . Prove that  $KA : KB = FA : FB$ .

46. Divide a given finite straight line similarly to a given divided straight line parallel to the first line.

47. If two parallel straight lines  $AB, CD$  be divided proportionally at  $P, Q$ , so that  $AP$  is to  $PB$  as  $CQ$  to  $QD$ , then the straight lines  $AC, PQ, BD$  meet in a point.

48.  $BAC, DAE$  are similar equal triangles,  $BAD$  and  $CAE$  being straight lines; and the parallelograms of which  $BC$  and  $DE$  are diagonals are completed. Prove that the lines drawn to complete the parallelograms themselves form a parallelogram whose diagonal passes through  $A$ .

49. If, in similar triangles, any two equal angles be joined to the opposite sides by straight lines making equal angles with homologous sides; these lines shall have the same ratio as the sides on which they fall, and shall divide those sides proportionally.

50.  $APB$ ,  $CQD$  are parallel straight lines, and  $AP$  is to  $PB$  as  $DQ$  to  $QC$ , prove that the straight lines  $PQ$ ,  $AD$ ,  $BC$  meet in a point.

51. Describe a circle which shall pass through a given point and touch two given straight lines.

52.  $AD$  the bisector of the base of the triangle  $ABC$  is bisected in  $E$ ,  $BE$  cuts  $AC$  in  $F$ , prove that  $AF : FC :: 1 : 2$ .

53. A straight line drawn through the middle point of a side of a triangle divides the other sides, one internally, the other externally in the same ratio.

54. In the triangle  $ABC$  there are drawn  $AD$  bisecting  $BC$ , and  $EF$  parallel to  $BC$  and cutting  $AB$ ,  $AC$  in  $E$ ,  $F$ . Shew that  $BF$  and  $CE$  intersect in  $AD$ .

55. In the triangle  $ACB$ , having  $C$  a right angle,  $AD$  bisecting the angle  $A$  meets  $CB$  in  $D$ , prove that the square on  $AC$  is to the square on  $AD$  as  $BC$  to  $2BD$ .

56.  $AB$  and  $CD$  are two parallel straight lines;  $E$  is the middle point of  $CD$ ;  $AC$  and  $BE$  meet at  $F$ , and  $AE$  and  $BD$  meet at  $G$ : shew that  $FG$  is parallel to  $AB$ .

57.  $A$ ,  $B$ ,  $C$  are three fixed points in a straight line; any straight line is drawn through  $C$ ; shew that the perpendiculars on it from  $A$  and  $B$  are in a constant ratio.

58. If the perpendiculars from two fixed points on a straight line passing between them be in a given ratio, the straight line must pass through a third fixed point.

59. Through a given point draw a straight line, so that the parts of it intercepted between that point and perpendiculars drawn to the straight line from two other given points may have a given ratio.

60. A tangent to a circle at the point  $A$  intersects two parallel tangents at  $B$ ,  $C$ , the points of contact of which with the circle are  $D$ ,  $E$  respectively; and  $BE$ ,  $CD$  intersect at  $F$ : shew that  $AF$  is parallel to the tangents  $BD$ ,  $CE$ .

61.  $P$  and  $Q$  are fixed points;  $AB$  and  $CD$  are fixed parallel straight lines; any straight line is drawn from  $P$  to meet  $AB$  at  $M$ , and a straight line is drawn from  $Q$  parallel to  $PM$  meeting  $CD$  at  $N$ : shew that the ratio of  $PM$  to  $QN$  is constant, and thence shew that the straight line through  $M$  and  $N$  passes through a fixed point.

62. If two circles touch each other, and also touch a given straight line, the part of the straight line between the points of contact is a mean proportional between the diameters of the circles.

63. If at a given point two circles intersect, and their centres lie on two fixed straight lines which pass through that point, shew that whatever be the magnitude of the circles their common tangents will always meet in one of two fixed straight lines which pass through the given point.

64. From the angular points of a parallelogram  $ABCD$  perpendiculars are drawn on the diagonals meeting them at  $E, F, G, H$  respectively: shew that  $EFGH$  is a parallelogram similar to  $ABCD$ .

65.  $ABCDE$  is a regular pentagon, and  $AD, BE$  intersect at  $O$ : shew that a side of the pentagon is a mean proportional between  $AO$  and  $AD$ .

66.  $ACB$  is a triangle, and the side  $AC$  is produced to  $D$  so that  $CD$  is equal to  $AC$ , and  $BD$  is joined: if any straight line drawn parallel to  $AB$  cuts the sides  $AC, CB$ , and from the points of section straight lines be drawn parallel to  $DB$ , shew that these straight lines will meet  $AB$  at points equidistant from its extremities.

67. If a circle be described touching externally two given circles, the straight line passing through the points of contact will intersect the straight line passing through the centres of the given circles at a fixed point.

68.  $A$  and  $B$  are two points on the circumference of a circle of which  $C$  is the centre; draw tangents at  $A$  and  $B$  meeting at  $T$ ; and from  $A$  draw  $AN$  perpendicular to  $CB$ : shew that  $BT$  is to  $BC$  as  $BN$  is to  $NA$ .

69. Find a point the perpendiculars from which on the sides of a given triangle shall be in a given ratio.

70. A quadrilateral  $ABCD$  is inscribed in a circle and its diagonals  $AC, BD$  meet at  $O$ . Points  $P, Q$  are taken in  $AB, CD$  such that  $AP$  is to  $PB$  as  $AO$  to  $OB$ , and  $CQ$  is to  $QD$  as  $CO$  to  $OD$ ; prove that a circle can be described to touch  $AB, CD$  at  $P, Q$ .

71. Prove that the diagonals of the complements of parallelograms about a diagonal of a parallelogram meet in the diagonal of the parallelogram.

72. Through a point  $G$  of the side  $CD$  of a parallelogram  $ABCD$  are drawn  $AG$  and  $BG$  meeting the sides in  $E$  and  $F$ ; and  $GH$  is drawn parallel to  $EF$ , meeting  $AF$  in  $H$ ; prove that  $FH$  is equal to  $AD$ .

73. Any regular polygon inscribed in a circle is a mean proportional between the inscribed and circumscribed regular polygons of half the number of sides.

74. If two sides of a parallelogram inscribed in a quadrilateral be parallel to one of the diagonals of the quadrilateral, then the other sides of the parallelogram are parallel to the other diagonal.

75. A circle is described round an equilateral triangle, and from any point in the circumference straight lines are drawn to the angular points of the triangle: shew that one of these straight lines is equal to the other two together.

76.  $ABC$  is a triangle. At  $A$  a straight line  $AD$  is drawn making the angle  $CAD$  equal to  $CBA$ , and at  $C$  the straight line  $CD$  is drawn making the angle  $ACD$  equal to  $BAC$ . Shew that  $AD$  is a fourth proportional to  $AB$ ,  $BC$  and  $CA$ .

77. If  $bd$  be drawn cutting the sides  $AB$ ,  $AD$  and the diagonal  $AC$  of the parallelogram  $ABCD$  in  $b$ ,  $d$ , and  $c$  respectively, so that  $Ab$  is equal to  $Ad$ , then the sum of  $AB$ ,  $AD$  is to the sum of  $Ab$ ,  $Ad$  as  $AC$  to twice  $Ac$ .

78. Having given the base of a triangle and the opposite angle, find that triangle for which the rectangle contained by the perpendiculars from the ends of the base on the opposite sides is greater than for any other.

79. Through each angular point of a quadrilateral a straight line is drawn perpendicular to the diagonal which does not pass through that point, shew that the parallelogram thus formed is similar to the parallelogram formed by joining the middle points of the sides of the given quadrilateral.

80.  $ABCD$  is a quadrilateral inscribed in a circle;  $BA$ ,  $CD$  produced meet in  $P$ , and  $AD$ ,  $BC$  produced in  $Q$ . Prove that  $PC$  is to  $PB$  as  $QA$  to  $QB$ .

81. Through  $D$ , any point in the base of a given triangle  $ABC$ , straight lines  $DE$ ,  $DF$  are drawn parallel to the sides  $AB$ ,  $AC$  and meeting the sides in  $E$ ,  $F$  and  $EF$  is drawn; shew that the triangle  $AEF$  is a mean proportional between the triangles  $FBD$ ,  $EDC$ .

82. If through the vertex and the extremities of the base of a triangle two circles be described intersecting each other in the base or the base produced, their diameters are proportional to the sides of the triangle.

83. Draw a straight line such that the perpendiculars let fall from any point in it on two given straight lines may be in a given ratio.

84. In any right-angled triangle, one side is to the other, as the excess of the hypotenuse above the second, to the line cut off from the first between the right angle and the line bisecting the opposite angle.

85.  $AB$  is a fixed straight line,  $C$  a moving point, and  $CD$  a line parallel to  $AB$ ; a variable line  $PQR$  is drawn cutting  $AC$  in  $P$ ,  $BC$  in  $Q$  and  $CD$  in  $R$ ; prove that if the ratios  $AP$  to  $PC$ , and  $BQ$  to  $QC$  be constant,  $CR$  is of constant length.

86. If  $I$ ,  $I_1$  be the centres of the inscribed circle of a triangle  $ABC$  and of the circle escribed beyond  $BC$ , the rectangle  $AI$ ,  $AI_1$  is equal to the rectangle  $AB$ ,  $AC$ .

87. If  $I$ ,  $I_1$  be the centres of the inscribed circle of a triangle  $ABC$  and of the circle escribed beyond  $BC$ , and  $D$ ,  $E$  be the points of contact of those circles with  $AB$ , then  $ID$  is to  $DB$  as  $EB$  to  $EI_1$ .

88. If  $I_1, I_2$  be the centres of the circles of a triangle  $ABC$  escribed beyond  $BC, CA$  respectively and  $E, F$  be their points of contact with  $AB$ , then  $I_1E$  is to  $EB$  as  $BF$  to  $I_2F$ .

89.  $O$  is a fixed point in a given straight line  $OA$ , and a circle of given radius moves so as always to be touched by  $OA$ ; a tangent  $OP$  is drawn from  $O$  to the circle, and in  $OP$  produced  $PQ$  is taken a third proportional to  $OP$  and the radius: shew that as the circle moves along  $OA$ , the point  $Q$  will move in a straight line.

90. On  $AB, AC$ , two adjacent sides of a rectangle, two similar triangles are constructed, and perpendiculars are drawn to  $AB, AC$  from the angles which they subtend, intersecting at the point  $P$ . If  $AB, AC$  be homologous sides, shew that  $P$  is in all cases in one of the diagonals of the rectangle.

91. If at any two points  $A, B$ ;  $AC, BD$  be drawn at right angles to  $AB$  on the same side of it, so that  $AB$  is a mean proportional between  $AC$  and  $BD$ ; the circles described on  $AC, BD$  as diameters will touch each other.

92. One circle touches another internally at  $A$ , and from two points in the line joining their centres, perpendiculars are drawn intersecting the outer circle in the points  $B, C$ , and the inner in  $D, E$ ; shew that  $AB : AC = AD : AE$ .

93. Find a straight line such that the perpendiculars on it from three given points shall be in given ratios to each other.

94. Divide a given arc of a circle into two parts, so that the chords of these parts shall be to each other in a given ratio.

95.  $CAB, CEB$  are two triangles having a common angle  $CBA$ , and the sides opposite to it  $CA, CE$  equal. If  $BA$  be produced to  $D$ , and  $ED$  be taken a third proportional to  $BA, AC$ , then the triangle  $BDC$  is similar to the triangle  $BAC$ .

96. One side of a triangle is given, and also its points of intersection with the bisector of the opposite angle and the perpendicular from the opposite vertex; construct the triangle.

97. The diameter of a circle is a mean proportional between the sides of an equilateral triangle and a regular hexagon which are described about the circle.

98. If two regular polygons of the same number of sides be constructed, one inscribed in and the other described about a given circle, and a third of double the number of sides be inscribed in the circle, this last is a mean proportional between the other two.

99. A triangle  $DEF$  is inscribed in a triangle  $ABC$  so that  $DE, DF$  are parallel to  $BA, CA$  respectively; prove that the triangle  $DEF$  is to the triangle  $ABC$  as the rectangle  $BD, DC$  to the square on  $BC$ .

100. The vertical angle  $C$  of a triangle is bisected by a straight line which meets the base at  $D$ , and is produced to meet the circle  $ABC$  at  $E$ ; prove that the rectangle contained by  $CD$  and  $CE$  is equal to the rectangle contained by  $AC$  and  $CB$ .

101. A straight line is divided in two given points, determine a third point, such that its distances from the two given points may be proportional to its distances from the ends of the line.

102.  $AB$  is a diameter of a circle,  $PQ$  a chord perpendicular to  $AB$ ,  $O$  any point on the circle;  $OP$ ,  $OQ$  meet  $AB$  in  $R$  and  $S$ ; prove that the rectangle  $AR \cdot BS$  is equal to the rectangle  $AS \cdot BR$ .

103.  $A$ ,  $B$  are two fixed points and  $P$  a variable point on a circle,  $AA'$ ,  $BB'$  are drawn parallel to a fixed line to meet the circle in  $A'$ ,  $B'$ : the fixed line meets  $AB'$  in  $D$ ,  $A'B$  in  $D'$ ,  $AP$  in  $E$ ,  $BP$  in  $E'$ ; prove that  $DE \cdot D'E'$  is constant.

104. Prove that, if  $ABCD$  be a quadrilateral not inscriptible in a circle, a point  $E$  exists such that the rectangle  $AB, CD$  is equal to the rectangle  $AE, BD$  and the rectangle  $AD, BC$  is equal to the rectangle  $CE, BD$ . Hence prove the converse of Ptolemy's Theorem.

105.  $BE$  and  $CF$  are perpendiculars upon  $AD$  the bisector of the angle  $A$  of a triangle  $ABC$ . The area of the triangle is equal to either of the rectangles  $AE, CF$  or  $AF, BE$ .

106. If the exterior angle  $CAE$  of a triangle be bisected by the straight line  $AD$  which likewise cuts the base produced in  $D$ ; then  $BA \cdot AC$ , the rectangle of the sides, is less than the rectangle  $BD \cdot DC$  by the square on  $AD$ .

107.  $ABC$  is an isosceles triangle, the side  $AB$  being equal to  $AC$ ;  $F$  is the middle point of  $BC$ ; on any straight line through  $A$  perpendiculars  $FG$  and  $CE$  are drawn: shew that the rectangle  $AC, EF$  is equal to the sum of the rectangles  $FC, EG$  and  $FA, FG$ .

108. Describe a circle which shall pass through a given point and touch a given straight line and a given circle.

109. Divide a triangle into two equal parts by a straight line at right angles to one of the sides.

110. If a straight line drawn from the vertex of an isosceles triangle to the base, be produced to meet the circumference of a circle described about the triangle, the rectangle contained by the whole line so produced and the part of it between the vertex and the base shall be equal to the square on either of the equal sides of the triangle.

111. Two straight lines are drawn from a point  $A$  to touch a circle of which the centre is  $E$ ; the points of contact are joined by a straight line which cuts  $EA$  at  $H$ ; and on  $HA$  as diameter a circle is described: shew that the straight lines drawn through  $E$  to touch this circle will meet it on the circumference of the given circle.

112. Two triangles  $BAD$ ,  $BAC$  have the side  $BA$  and the angle  $A$  common: moreover the angle  $ABD$  is equal to the angle  $ACB$ : shew that the rectangle contained by  $AC$ ,  $BD$  is equal to that contained by  $AB$ ,  $BC$ .

113.  $ABCD$  is a quadrilateral in a circle; the straight lines  $CE$ ,  $DE$  which bisect the angles  $ACB$ ,  $ADB$ , cut  $BD$  and  $AC$  at  $F$  and  $G$  respectively: shew that  $EF$  is to  $EG$  as  $ED$  is to  $EC$ .

114. A square  $DEFG$  is inscribed in a right-angled triangle  $ABC$ , so that  $D$ ,  $G$  are in the hypotenuse  $AB$  of the triangle  $E$  in  $AC$ , and  $F$  in  $CB$ : prove that the area of the square is equal to the rectangle  $AD$ ,  $BG$ .

115.  $A$ ,  $B$ ,  $C$  are three points in order in a straight line: find a point  $P$  in the straight line so that  $PB$  may be a mean proportional between  $PA$  and  $PC$ .

116.  $AB$  is a diameter, and  $P$  any point in the circumference of a circle;  $AP$  and  $BP$  are joined and produced if necessary; from any point  $C$  in  $AB$  a straight line is drawn at right angles to  $AB$  meeting  $AP$  at  $D$  and  $BP$  at  $E$ , and the circumference of the circle at  $F$ : shew that  $CD$  is a third proportional to  $CE$  and  $CF$ .

117. If  $F$  be a point in the side  $CB$  of a right-angled triangle and  $CD$ ,  $FE$  be perpendiculars on the hypotenuse  $AB$ , then the sum of the rectangles  $AD$ ,  $AE$  and  $CD$ ,  $EF$  is equal to the square on  $AC$ .

118. In the figure of II. 11 shew that four other straight lines besides the given straight line are divided in the required manner.

119. A straight line  $CD$  bisects the vertical angle  $C$  of a triangle  $ABC$ , and cuts the base in  $D$ , on  $AB$  produced a point  $E$  is taken equidistant from  $C$  and  $D$ : prove that the rectangle  $AE$ ,  $BE$  is equal to the square on  $DE$ .

120. If the perpendicular in a right-angled triangle divide the hypotenuse in extreme and mean ratio, the less side is equal to the alternate segment.

121.  $ABC$  is a right-angled triangle,  $CD$  a perpendicular from the right angle upon  $AB$ ; shew that if  $AC$  is double of  $BC$ ,  $BD$  is one-fifth of  $AB$ .

122. Through a given point draw a chord in a given circle so that it shall be divided at the point in a given ratio. Find the limiting value of the ratio.

123.  $ABCD$  is a parallelogram; from  $B$  a straight line is drawn cutting the diagonal  $AC$  at  $F$ , the side  $DC$  at  $G$ , and the side  $AD$  produced at  $E$ : shew that the rectangle  $EF$ ,  $FG$  is equal to the square on  $BF$ .

124. Find a point in a side of a triangle from which two straight lines drawn, one to the opposite angle, and the other parallel to the base, shall cut off towards the vertex and towards the base, equal triangles.

125. On a chord  $AB$  of a circle any point  $P$  is taken: on  $AP$ ,  $PB$  any two similar and similarly situated triangles  $APE$ ,  $PBF$  are constructed, and the straight line  $EF$  joining the vertices of these triangles is produced to meet  $AB$  produced in  $Q$ . If any circle be described touching  $AB$  at  $P$  the common chord of these two circles passes through  $Q$ .

126. With a point  $A$  in the circumference of a circle  $ABC$  as centre, a circle  $PBC$  is described cutting the former circle at the points  $B$  and  $C$ ; any chord  $AD$  of the former meets the common chord  $BC$  at  $E$ , and the circumference of the other circle at  $O$ : shew that the angles  $EPO$  and  $DPO$  are equal for all positions of  $P$ .

127. It is required to cut off from one given line a part such that it may be a mean proportional between the remainder and another given line.

128. Construct a square so that its vertices shall lie on four of the sides of a regular pentagon.

129. Shew how to divide a given triangle into any number of equal parts by lines parallel to the base.

130. Divide a given triangle by a straight line drawn in a given direction into two parts whose areas shall be in a given ratio.

131. If  $E$  be the intersection of the diagonals of a quadrilateral  $ABCD$ , which has the sides  $AB$  and  $CD$  parallel, then

(i) the straight line joining the middle points of  $AB$  and  $CD$  passes through  $E$ ;

(ii) if  $P$  be any point in  $DB$  produced, the rectangles  $PB$ ,  $EC$  and  $PD$ ,  $EA$  are together equal to the rectangle  $PE$ ,  $AC$ .

132. Two quadrilaterals  $ABCD$ ,  $ABEF$  in which  $BC$ ,  $CD$ ,  $DA$  are equal to  $BE$ ,  $EF$ ,  $FA$  respectively, are on the same side of  $AB$ . Prove that if the rectangles  $OA$ ,  $OD$  and  $OB$ ,  $OC$  be equal, where  $O$  is the point of intersection of the bisectors of the angles,  $DAF$ ,  $CBE$ , then the quadrilaterals are equal in area.

133. Prove that the area of a quadrilateral, whose sides are all of given lengths, is a maximum when two opposite angles of the quadrilateral are supplementary.

134. Having four given finite straight lines, construct the quadrilateral of maximum area which can be formed with them taken in a given order for sides.

135. Either of the complements is a mean proportional between the parallelograms about the diameter of a parallelogram.

136. Shew that one of the triangles in the figure of iv. 10 is a mean proportional between the other two.

137. The sides  $AB$  and  $AC$  of a triangle  $ABC$  are produced to  $D$  and  $E$  respectively, and  $DE$  is joined. A point  $F$  is taken in  $BC$  such that  $BF:FC=AB.AE:AC.AD$ , prove that, if  $AF$  be joined and produced, it will pass through the middle point of  $DE$ .



138. In any triangle  $ABC$ , if  $BD$  be taken equal to one-fourth of  $BC$ , and  $CE$  one-fourth of  $AC$ , the straight line drawn from  $C$  through the intersection of  $BE$  and  $AD$  will divide  $AB$  into two parts, which are in the ratio of nine to one.

139. Lines drawn from the extremities of the base of a triangle intersecting on the line joining the vertex with the middle point of the base, cut the sides proportionally; and conversely.

140.  $D$  is the middle point of the side  $BC$  of a triangle  $ABC$ , and  $P$  is any point in  $AD$ ; through  $P$  the straight lines  $BPE$ ,  $CPF$  are drawn meeting the other sides at  $E$ ,  $F$ : shew that  $EF$  is parallel to  $BC$ .

141.  $ABC$  is a triangle and  $D$ ,  $E$ ,  $F$  points in the sides  $BC$ ,  $CA$ ,  $AB$  respectively such that  $AD$ ,  $BE$ ,  $CF$  meet in  $O$ ; prove that the ratio  $AO$  to  $OD$  is equal to the sum of the ratios  $AF$  to  $FB$  and  $AE$  to  $EC$ .

142. Through any point  $O$  within triangle  $ABC$  straight lines  $AO$ ,  $BO$ ,  $CO$  are drawn cutting the opposite sides in  $D$ ,  $E$ ,  $F$ .  $EF$ ,  $FD$ ,  $DE$  are produced to meet the sides again in  $G$ ,  $H$ ,  $K$ . Prove that circles on  $DG$ ,  $EH$ ,  $FK$  as diameters pass through the same two points.

143. Find a point without two circles such that the tangents drawn therefrom to the circles shall contain equal angles.

144. Prove that the locus of a point, at which two given parts of the same straight line subtend equal angles is two circles.

145. Find on a given straight line  $AB$  two points  $P$ ,  $Q$  such that  $APQB$  is a harmonic range, and the ratio  $AP$  to  $PQ$  is equal to a given ratio.

146.  $P$  is a point on the same straight line as the harmonic range  $ABCD$ ; prove that

$$2 \frac{PA}{AC} = \frac{PB}{BC} + \frac{PD}{DC}.$$

147. A chord  $AB$  and a diameter  $CD$  of a circle cut at right angles. If  $P$  be any other point on the circle,  $P(ACBD)$  is a harmonic pencil.

148. If two circles touch one another externally and from the centre of one tangents be drawn to the other, the chord joining the points in which the first circle is cut by the tangents, will be an harmonic mean between the radii.

149.  $A$ ,  $B$ ,  $C$ ,  $A'$ ,  $B'$ ,  $C'$  are two sets of three points lying on two parallel straight lines; prove that the intersections of the three pairs of lines  $AA'$ ,  $BB'$ ;  $BB'$ ,  $CC'$ ;  $CC'$ ,  $AA'$  lie on a straight line.

150.  $P$  and  $Q$  are any two points in  $AD$ ,  $BC$  two opposite sides of a parallelogram;  $X$  and  $Y$  are the respective intersections of  $AQ$ ,  $BP$ , and  $DQ$ ,  $CP$ ; prove that  $XY$ , produced, bisects the parallelogram.

151. If the sides of a quadrilateral circumscribing a circle touch at the angular points of an inscribed quadrilateral, all the diagonals meet in a point.

152. The square inscribed in a circle is to the square inscribed in the semicircle as 5 to 2.

153. Describe a rectangle which shall be equal to a given square and have its sides in a given ratio.

154. In a given rectangle inscribe a rectangle whose sides shall have to one another a given ratio.

155. Describe a triangle similar to a given triangle, one angular point being on the bounding diameter of a given semicircle, and one of the sides perpendicular to this diameter, and the other two angular points on the arc of the semicircle.

156. If  $M$ ,  $N$  be the points at which the inscribed and an escribed circle touch the side  $AC$  of a triangle  $ABC$  and if  $BM$  be produced to cut the escribed circle again at  $P$ , then  $NP$  is a diameter.

157. Shew that in general two circles can be described to cut two lines  $AB$ ,  $AC$  at given angles and to pass through a fixed point  $P$ . If  $T$ ,  $T'$  be the centres of these circles, then  $PA$  bisects the exterior angle  $TCT'$ .

158. From a given point outside two given circles which do not meet, draw a straight line such that the portions of it intercepted by the circles shall be in the same ratio as their radii.

159.  $A'$ ,  $B'$ ,  $C'$  are the middle points of the sides of the triangle  $ABC$ , and  $P$  is the orthocentre and  $PA'$ ,  $PB'$ ,  $PC'$  produced meet the circumscribing circle in  $A''$ ,  $B''$ ,  $C''$ ; prove that the triangle  $A''B''C''$  is equal in all respects to the triangle  $ABC$ .

160. If through any point in the arc of a quadrant, whose radius is  $R$ , two circles be drawn touching the bounding radii of the quadrant, and  $r$ ,  $r'$  be the radii of these circles, then  $rr' = R^2$ .

161. Let two circles touch one another, and a common tangent be drawn to them touching them in  $P$ ,  $Q$ . If a pair of parallel tangents be drawn to the two circles meeting  $PQ$  in  $A$ ,  $B$ , and if the line joining their points of contact meet  $PQ$  in  $C$ , then the ratio of  $AP$  to  $BQ$  is either equal to or duplicate of the ratio of  $PC$  to  $QC$ , according as one or another pair of parallel tangents is taken.

162. If three circles touch a straight line one of the circles which touches the three circles passes through their radical centre.

163. Two circles cut in the points  $A$ ,  $B$ ; any chord through  $A$  cuts the circles again at  $P$  and  $Q$ ; shew that the locus of the point dividing  $PQ$  in a constant ratio is a circle passing through  $A$  and  $B$ .

164.  $AB$  and  $AC$  are two fixed straight lines, and  $O$  is a fixed point. Two circles are drawn through  $O$ , one of which touches  $AB$  and  $AC$  at  $D$  and  $E$  respectively, and the other touches them at  $F$  and  $G$  respectively. Prove that the circles passing round  $ODF$  and  $OEG$  touch one another at  $O$ .

165. Prove that the locus of the middle points of the sides of all triangles which have a given orthocentre and are inscribed in a given circle is another circle.

166. From any point  $P$  on the circle  $ABC$  a pair of tangents  $PQ, PR$  are drawn to the circle  $DEF$ , and the chord  $QR$  is bisected in  $S$ . Shew that the locus of  $S$  is a circle; except when the circle  $ABC$  passes through the centre of the circle  $DEF$ , when the locus of  $S$  is a straight line.

167. It is required to describe a circle through two given points  $A, B$  and to touch a given circle which touches  $AB$  at  $D$ . Find  $C$  the centre of the circle: draw  $CA, CB$ , and draw  $AO, BO$  at right angles to  $CA, CB$  respectively. Prove that  $OD$  produced will cut the circle in a point  $P$ , such that the circle  $APB$  will touch the given circle at  $P$ .

168.  $O$  is the radical centre of three circles. Points  $A, B, C$  are taken on the radical axes and  $AB, BC, CA$  are drawn. Prove that the six points in which these meet the three given circles lie on a circle. If radii vectores are drawn from  $O$  to these six points, they meet the three given circles in six points on a circle and its common chords with the three circles meet in pairs on  $OA, OB, OC$ .

169. If from a given point  $S$ , a perpendicular  $SY$  be drawn to the tangent  $PY$  at any point  $P$  of a circle, centre  $C$ , and in the line  $MP$  drawn perpendicular to  $CS$ , or in  $MP$  produced, a point  $Q$  be taken so that  $MQ = SY$ ,  $Q$  will lie on one of two fixed straight lines.

170. The diagonals  $AC, BD$ , of a quadrilateral inscribed in a circle cut each other in  $E$ . Shew that the rectangle  $AB, BC$  is to the rectangle  $AD, DC$  as  $BE$  to  $ED$ .

171. The square on the straight line joining the centres of the circumscribed circle and an escribed circle of a triangle is greater than the square on the radius of the circumscribed circle by twice the rectangle contained by the radii of the circles. (See p. 476.)

172. If one triangle can be constructed such that one of two given circles is the circumscribed circle and the other of the given circles is one of its escribed circles, an infinite number of such triangles can be constructed. (See p. 477.)

173.  $A, B, C$  are three circles: prove that, if the common tangent of  $A$  and  $C$  be equal to the sum of the common tangents of  $A$  and  $B$  and of  $B$  and  $C$ , the three circles touch a straight line.

174. Three circles  $A, B, C$  touch a fourth circle: prove that the ratio of the common tangent of  $A$  and  $B$  to the common tangent of  $A$  and  $C$  is independent of the radius of  $A$ .

175. If  $ABCD$  be a quadrilateral inscribed in a circle and a second circle touch this at  $A$ , and the tangents to it from  $B, C, D$  be  $Bb, Cc, Dd$ , then the rectangle  $BD, Cc$  is equal to the sum of the rectangles  $BC, Dd$  and  $CD, Bb$ .

176.  $ABCD$  is a quadrilateral inscribed in a circle,  $F$  the intersection of the diagonals: shew that the rectangle  $AF, FD$  is to the rectangle  $BF, FC$  as the square on  $AD$  to the square on  $BC$ .

177. The diagonals of a quadrilateral, inscribed in a circle, are to one another in the same ratio as the sums of the rectangles contained by the sides which meet their extremities.

178. The sides of a quadrilateral  $ABCD$  produced meet in  $P$  and  $Q$ . Prove that the rectangles  $PD, DQ; AD, DC; PB, BQ; AB, BC$  are proportionals.

179. Prove that the locus of a point such that the square on its distance from a fixed point varies as its distance from a fixed straight line is a circle.

180. A quadrilateral circumscribes a circle. Shew that the rectangles contained by perpendiculars from opposite angles upon any tangent are to one another in a constant ratio.

## BOOK XI.

### DEFINITIONS.

THE Geometrical propositions in the preceding Books of Euclid relate to figures in one plane. The propositions of this Book relate to figures which are not entirely confined to one plane. This part of Geometry is called Solid Geometry.

It must be remembered that the diagrams throughout this Book are intended to represent figures which do not lie in one plane. To more readily suggest the relative position of the lines in the figure, a line is dotted when it is supposed to be seen through part of a plane which constitutes part of the figure. Thick lines are sometimes introduced in the diagrams with a view of more readily suggesting to the eye the relative positions of different parts of the figure.

The reader must be warned that equal straight lines and equal angles in the figure are represented in the diagrams by straight lines and angles which are not necessarily equal and in most cases are unequal. The principles on which the diagrams are drawn will be fully understood when the reader has acquired a knowledge of the principles of Perspective Drawing (see pages 576—581).

The definition of a plane, adopted in Book I. (page 6), was

\* *A surface such that the straight line joining any two points in the surface lies wholly in the surface is called a plane.*

It follows at once from this definition and from Postulates 1 and 2 of Book I. that:—

(i) *One part of a straight line cannot lie in a plane and another part without the plane;*

(ii) *If two planes have two points in common, the straight line joining the points lies wholly in each of the planes;*

(iii) *If two planes have three points, which do not lie in a straight line, in common, the planes are coincident throughout their whole surface.*

The first two results are obvious. The third may be proved thus:

Let  $A, B, C$  be the three points, which are common to the two planes; draw the straight lines  $AB, BC, CA$ . These lines lie wholly in each of the two planes. Any straight line in one of the planes must intersect at least two of the lines  $AB, BC, CA$ : let it intersect two of them in  $P, Q$ : then since  $P, Q$  are also in the second plane, the whole line  $PQ$  lies in the second plane. Hence every straight line in either of the planes lies wholly also in the other, and therefore the planes are coincident throughout their whole surface.

*A straight line, which lies wholly in each of two planes is often called the common section of the planes.*

POSTULATE 1. *A plane may be drawn through any three points.*

It appears from what has been said already that, if the three given points do not lie in a straight line, only one plane can be drawn through them: but that, if they do lie in a straight line, a plane can be drawn through them and any other point chosen arbitrarily.

POSTULATE 2. *A part of a plane may be produced to any extent in any direction in its plane.*

**DEFINITION 1.** *The angular opening between two or more planes, at a point where they meet, is called a solid angle or a polyhedral\* angle.*

*A solid angle formed by two planes is called dihedral\*,  
one formed by three planes is called trihedral\*,  
one formed by four planes is called tetrahedral\*,  
one formed by five planes is called pentahedral\*.*

Names might be given for a solid angle at a point where six or more planes meet, but such names would be but seldom used.

The student must be warned here that solid angles and plane angles are geometrical magnitudes of essentially different kinds. The Sixth Book has taught him that plane angles are proportional to the arcs of equal circles which subtend them at their centres. In the same way it can be proved that solid angles are proportional to the areas of the parts of equal spheres which subtend them at the centres. When this is proved, it appears that, whereas a plane angle may be measured by the *length* of the arc of a circle, a solid angle may be measured by the *area* of part of the surface of a sphere.

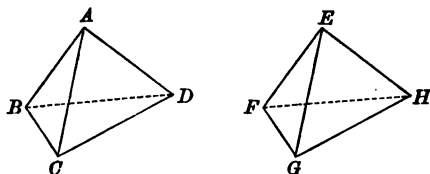
It must also be noted that, just as the magnitude of a plane angle is not affected by the lengths of the straight lines which bound it, so the magnitude of a solid angle is not affected by the distance to which the bounding planes extend from the point at which the solid angle exists.

It is necessary to have some method of denoting the solid angle at a point contained by given planes. It is desirable to select a system which is applicable to solid angles contained by two or any greater number of planes. The method which is used in this work is to write first the letter denoting the point at which the solid angle exists, and afterwards letters to denote any surface, which subtends the angle at the point. For instance the trihedral angle at the point  $O$  contained by the planes  $AOB$ ,  $BOC$ ,  $COA$  is denoted by  $O(ABC)$ , and the tetrahedral angle contained at the point  $O$  by four planes  $AOB$ ,  $BOC$ ,  $COD$ ,  $DOA$  by  $O(ABCD)$ .

\* Derived from πολὺς "much," δύο "two," τρεῖς "three," τέτταρες "four," πέντε "five" respectively, and ἑδρα "a base."

In applying this system to a dihedral angle we mention two points in the common section of the planes which form the angle, and also a point in each of the planes: for instance, the dihedral angle contained by two planes  $ABC$ ,  $ABD$  at a point  $O$  in  $AB$  is denoted by  $O(ACBD)$ .

**Test of Equality of Solid Angles.** Strictly speaking two solid angles are said to be equal, when it is possible to shift the planes forming one of the angles, unchanged in position relative to each other, so as to exactly coincide in direction with the planes forming the other solid angle.



For instance, the trihedral angles at  $A$  and  $E$  will be equal, if it be possible to shift the planes  $BAC$ ,  $CAD$ ,  $DAB$  unchanged in position relative to each other, so that  $A$  coincides with  $E$  and each of the planes  $BAC$ ,  $CAD$ ,  $DAB$  coincides in direction with one of the planes  $FEG$ ,  $GEH$ ,  $HEF$ : it is clear that this is the same as saying the trihedral angles at  $A$  and  $E$  will be equal, if it be possible to shift the planes so that  $A$  coincides with  $E$  and each of the lines  $AB$ ,  $AC$ ,  $AD$  coincides in direction with one of the lines  $EF$ ,  $EG$ ,  $EH$ .

In Book I. (see page 93) two figures have been called equal when they were only *equal in area* and not *equal in all respects*. In the same way two solid angles will be sometimes called equal when they are equal in opening though not equal in all respects, that is when they are capable of being divided into a number of solid angles which in pairs are equal in all respects. When it is necessary to avoid all danger of mistake we shall therefore for the future in general speak of two solid angles which are equal in all respects as *equal in all respects* or as *equal and similar*; and we shall speak of two angles which are not equal in all respects, but which admit of division into solid angles which in pairs are equal in all respects,



as *equal in opening*. When however there is no danger of ambiguity, we shall speak of angles which are merely equal in opening as *equal*.

It may be remarked that equal and similar solid angles at the centres of equal spheres are subtended by portions of the surfaces of the spheres which are equal in all respects, but that solid angles which are only equal in opening are subtended by portions of the surfaces of the spheres which are only equal in area.

**Dihedral Angles.** If a plane move, while two points in the plane remain fixed in space, the plane is said to turn, rotate, or revolve about the straight line joining the fixed points. If the revolving plane move from any one position to any other position, it generates at any point in the line joining the fixed points a dihedral angle.

It will be proved (Prop. 10 Coroll. 2) that the dihedral angles between two planes at all points in their line of section are the same.

The two dihedral angles which two planes make at a point on the same side of one of the planes but on opposite sides of the other are called **adjacent dihedral angles**.

**DEFINITION 2.** *If two solid angles at a point be such that each is formed by producing the planes which contain the other through the point, they are called **vertically opposite solid angles**.*

**DEFINITION 3.** *If two adjacent dihedral angles made by two planes at a point be equal, the planes are said to be at **right angles** to each other.*

It will be proved (Prop. 10 Coroll. 2) that if two planes be at right angles at one point of their line of section they are at right angles at every point of it.

**DEFINITION 4.** *The trihedral angle made by three planes which are at right angles to each other in pairs is called a **right solid angle**.*

**DEFINITION 5.** *When a straight line is at right angles to every straight line meeting it in the plane, it is said to be at right angles to the plane, and the plane is said to be at right angles to the line.*

It will be proved hereafter (Prop. 4) that, if a straight line be at right angles to two straight lines meeting it in a plane, it is at right angles to every straight line meeting it in the plane.

**DEFINITION 6.** *Two planes, which do not meet however far they may be produced in every possible direction, are said to be parallel to one another.*

*A plane and a straight line, which do not meet however far they may be produced in every possible direction, are said to be parallel to one another.*

**DEFINITION 7.** *If two planes be not parallel to one another, they are said to be inclined to one another.*

*If from a point in the common section of two planes a straight line be drawn in each of the planes at right angles to their common section the angle between these lines is called the inclination of the planes to one another at the point.*

It will be proved (Prop. 10 Coroll. 1) that the inclination of two planes to one another is the same at every point of their section.

**DEFINITION 8.** *If a straight line be not parallel to a plane, it is said to be inclined to the plane.*

The smallest angle made by it with a straight line meeting it in the plane is called the inclination of the straight line to the plane.

**DEFINITION 9.** *A closed rectilineal figure, whose sides do not all lie in one plane, is called a skew figure.*

**DEFINITION 10.** *A surface, such that every section of it made by any plane which intersects it consists of one or more closed lines, is called a closed surface.*

*A closed surface composed wholly of planes is called a polyhedron\*.*

*The planes, which form a polyhedron, are called the faces of the figure.*

*The line, in which two faces meet, is called an edge of the figure.*

*The point, where three or more edges meet, is called a vertex or an angular point or a corner of the figure.*

*A polyhedron of four faces is called a tetrahedron\*,  
 one of five faces is called a pentahedron\*,  
 one of six faces is called a hexahedron\*,  
 one of eight faces is called an octahedron\*,  
 one of twelve faces is called a dodecahedron\*,  
 one of twenty faces is called an icosahedron\*.*

DEFINITION 11. *A polyhedron, which is such that the whole figure lies on one side only of each of the faces of the figure, is called a convex polyhedron.*

It is at once seen that every plane section of a convex polyhedron is a convex polygon.

DEFINITION 12. *A polyhedron, of which all the faces are regular polygons of the same kind and all the solid angles are equal, is called a regular solid.*

It will be proved hereafter that there are five and only five possible convex regular solids and that the tetrahedron, the hexahedron, the octahedron, the dodecahedron and the icosahedron are the only polyhedrons which can possibly be regular.

The regular hexahedron is generally called a cube. (See Def. 13.)

\* Derived from πολὺς "much," τέτταρες "four," πέντε "five," ἕξ "six," ὀκτώ "eight," δώδεκα "twelve," εἰκοσι "twenty" respectively, and ἑδρα "a base."

DEFINITION 13. *A hexahedron, whose opposite faces are parallel, is called a parallelepiped\*.*

*A parallelepiped, one of whose trihedral angles is a right solid angle, is called a rectangular parallelepiped, or an orthohedron†.*

It will be proved later that each trihedral angle of a rectangular parallelepiped is a right solid angle.

*An orthohedron, three of whose edges meeting in a vertex are equal, is called a cube‡.*

DEFINITION 14. *A polyhedron, all but one of whose faces meet in a point, is called a pyramid§.*

DEFINITION 15. *A polyhedron, all but two of the faces of which are parallel to one straight line, is called a prism||.*

The two faces of the prism, which are not parallel to the straight line, are generally assumed to be parallel to one another.

DEFINITION 16. *Two solid figures are said to be equal in all respects, when it is possible to shift one unchanged in shape and size so as to coincide with the other.*

DEFINITION 17. *When the solid angles of one polyhedron taken in order are equal and similar to the solid angles of another taken in order, the two polyhedrons are said to be equiangular to one another.*

*When two polyhedrons are equiangular to one another, and when the ratio of an edge of one of them to the corresponding edge of the other is the same for all pairs of corresponding edges, the polyhedrons are said to be similar to one another.*

\* Derived from παράλληλος "parallel," and ἐπίπεδον "a plane."

† Derived from ὀρθός "right (at right angles)," and ἔδρα "a base."

‡ Derived from κύβος "a die."

§ Derived from πυραμὶς "a pyramid."

|| Derived from πρίσμα "a prism."

**DEFINITION 18.** *A closed surface, which is such that all straight lines drawn to it from a fixed point are equal, is called a sphere\*.*

*This point is called the centre of the sphere.*

It will be proved (page 590) that a sphere has only one centre.

*A straight line drawn from the centre of a sphere to the sphere is called a radius.*

*A straight line drawn through the centre and terminated both ways by the sphere is called a diameter.*

It will be proved (page 573) that four points on a sphere not in the same plane completely fix the position and magnitude of the sphere: hence we generally denote a sphere by mentioning four points on it.

**DEFINITION 19.** *A straight line drawn through the centre of a circle at right angles to its plane is called the axis of the circle.*

**DEFINITION 20.** *If a straight line pass through a moveable point on a fixed circle and a fixed point not in the plane of the circle, the surface generated by the straight line, when the moveable point describes the circle, is called a circular cone†.*

*The fixed point is called the vertex of the cone.*

*If the fixed point lie on the axis of the circle the cone is said to be a right circular cone.*

**DEFINITION 21.** *If a straight line be drawn through a moveable point on a circle parallel to a fixed straight line, the surface generated by the moving line, as the moveable point describes the circle, is called a circular cylinder‡.*

*If the moving line be parallel to the axis of the circle, the cylinder is said to be a right circular cylinder.*

\* Derived from σφαῖρα "a ball."

† Derived from κώνος "a cone."

‡ Derived from κύλινδρος "a cylinder."

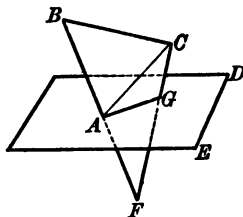
## PROPOSITION 1.

*If two planes meet in a point, they meet in a straight line.*

Let  $ABC$ ,  $ADE$  be two given planes, which meet at the point  $A$  :

it is required to prove that they meet in a straight line.

CONSTRUCTION. Take any two points  $B$ ,  $C$  lying in the plane  $ABC$ , and not in the plane  $ADE$  but on the same side of it; draw  $AB$ ,  $AC$  and produce  $BA$  to  $F$ . Draw  $CF$ .



PROOF. Because  $B$  and  $C$  are on the same side of the plane  $ADE$ ,

and  $B$  and  $F$  are on opposite sides,

therefore  $C$  and  $F$  are on opposite sides.

Therefore the straight line  $CF$  must meet the plane  $ADE$  in some point, say  $G$ .

Then because  $A$  and  $G$  are in each of the planes  $ABC$ ,  $ADE$ ,

the straight line  $AG$  is in both planes. (I. Def. 7.)

Wherefore, if two planes &c.

## EXERCISES.

1. How many different planes can be drawn through three out of (a) four, (b) five given points?
2. How many different straight lines are formed by the intersections of two out of (a) three, (b) four, (c) five planes?
3. Into how many parts is space divided by (a) one, (b) two, (c) three, (d) four planes?
4. Shew how to draw through a given point a straight line to intersect two non-intersecting straight lines.
5. Prove that every plane not passing through an angular point of a skew quadrilateral cuts an even number of the sides of the quadrilateral in points in the sides themselves and an even number in points in the sides produced.
6. If two triangles  $ABC$ ,  $A'B'C'$  in different planes be such that the lines  $AA'$ ,  $BB'$ ,  $CC'$  meet in a point  $S$ , then the three pairs of corresponding sides  $BC$ ,  $B'C'$ ;  $CA$ ,  $C'A'$ ;  $AB$ ,  $A'B'$  intersect in points lying on a straight line.

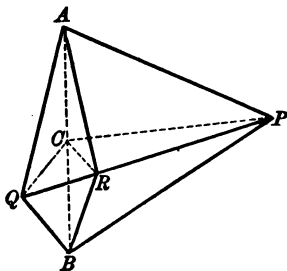
## PROPOSITION 2.

*Every point, which is equidistant from two fixed points, lies in a fixed plane.*

Let  $A, B$  be two given points:  
it is required to prove that every point which is equidistant from  $A$  and  $B$  lies in a fixed plane.

CONSTRUCTION. Take  $P, Q$  any two points, such that each is equidistant from  $A$  and  $B$ . Draw  $AB$  and bisect  $AB$  in  $C$  and draw  $CP, PQ, QC$ .

Take any point  $R$  in  $PQ$  and draw  $RA, RB, RC$ .



PROOF. Because in the triangles  $APQ, BPQ$ ,  
 $AP, PQ, QA$  are equal to  $BP, PQ, QB$  respectively,  
the angles  $APQ, BPQ$  are equal. (I. Prop. 8.)

And because in the triangles  $APR, BPR$

$AP, PR$  are equal to  $BP, PR$  respectively,  
and the angles  $APR, BPR$  are equal,

$RA$  is equal to  $RB$ ; (I. Prop. 4.)

that is, any point  $R$  in the straight line joining  $P, Q$  two points which are equidistant from  $A$  and  $B$

is also itself equidistant from  $A$  and  $B$ .

Similarly it can be proved that any point on  $CP$  or  $CQ$  satisfies the same condition.

Therefore every point in the plane  $CPQ$  is equidistant from  $A$  and  $B$ .

Again no other point can be equidistant: for if the straight line  $AS$ , joining  $A$  to any point  $S$  not in the plane, cut the plane in  $P$ ;



because  $PA$  is equal to  $PB$   
the angles  $PBA$ ,  $PAB$  are equal. (I. Prop. 5.)  
Therefore the angles  $SBA$ ,  $SAB$  are not equal,  
and  $SA$  is not equal to  $SB$ . (I. Prop. 19.)

Wherefore, *every point &c.*

### EXERCISES.

1. The plane which is the locus of a point equidistant from two given points bisects the straight line joining the points, and is at right angles to it.

2. The locus of a point which is equidistant from three given points is a straight line which passes through the centre of the circle through the given points.

3. The straight line which is the locus of a point equidistant from three given points is at right angles to the plane through the given points.

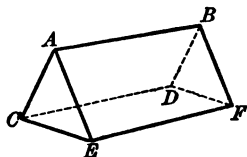
4. Prove that, if a straight line  $AP$  make equal angles with two given straight lines  $AB$ ,  $AC$ , it lies in one or other of two fixed planes.

5. Every point which is equidistant from two given intersecting straight lines lies in one or other of two fixed planes.

## PROPOSITION 3.

*If three planes intersect each other in pairs, their common sections either meet in a point or are parallel in pairs.*

Let  $ABDC$ ,  $CDFE$ ,  $EFBA$  be three planes which intersect each other in pairs in the straight lines  $EF$ ,  $AB$ ,  $CD$ : it is required to prove that  $AB$ ,  $CD$ ,  $EF$  either meet in a point or are parallel in pairs.



**PROOF.** First, if  $AB$ ,  $CD$  intersect:  
 since every point in  $AB$  is common to the planes  $ABDC$ ,  $EFBA$  and every point in  $CD$  is common to the planes  $ABDC$ ,  $CDFE$ ,  
 the intersection of  $AB$  and  $CD$  is common to the planes  $CDFE$ ,  $EFBA$ ;  
 that is to say the straight line  $EF$  passes through the intersection of  $AB$  and  $CD$ .

Secondly, if  $AB$ ,  $CD$  be parallel;  
 $EF$  cannot intersect  $AB$ , since, if it did, then, by the first case,  $CD$  would meet  $AB$  in the same point: but it is parallel to it;  
 and because  $AB$  and  $EF$  lie in the plane  $ABEF$  and they do not meet, they are parallel. (I. Def. 9.)  
 Similarly it can be proved that  $CD$  and  $EF$  are parallel.

Wherefore, if three planes &c.

## EXERCISES.

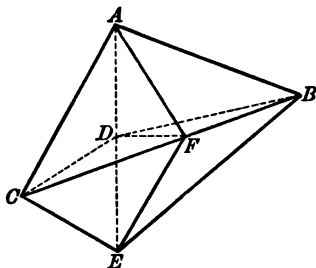
1. How many different points are formed by the intersection of three planes out of (a) four, (b) five given planes?
2. If three straight lines intersect each other in pairs, they meet at a point or lie in a plane.
3. If two straight lines, one in each of two intersecting planes, be parallel, each is parallel to the common section of the planes.
4. Draw through a given straight line a plane parallel to another given straight line.
5. Draw a straight line to meet two given non-intersecting straight lines and parallel to a third given straight line.  
Shew that the solution is, in general, unique.
6. If two straight lines be parallel, any plane through one of them is parallel to the other.
7. If a plane cut a tetrahedron in a parallelogram, the plane is parallel to two opposite edges of the tetrahedron.

## PROPOSITION 4.

*If one of three concurrent straight lines be at right angles to the other two, the first is at right angles to the plane through the others.*

Let  $DA, DB, DC$  be three concurrent straight lines such that the angles  $ADB, ADC$  are right angles: it is required to prove that  $DA$  is at right angles to the plane  $BDC$ .

CONSTRUCTION. Produce  $AD$  to  $E$  and make  $DE$  equal to  $DA$ . Draw  $AB, BE, AC, CE, CB$ . In the plane  $BDC$  draw through  $D$  any straight line  $DF$  meeting  $BC$  in the point  $F$  and draw  $FA, FE$ .



PROOF. Because in the triangles  $ABD, EBD$   
 $AD, DB$  are equal to  $ED, DB$  respectively,  
 and the angle  $ADB$  to the angle  $EDB$ ,  
 $AB$  is equal to  $EB$ . (I. Prop. 4.)

Similarly it can be proved that  
 $AC$  is equal to  $EC$ .

And because in the triangles  $ABC, EBC$ ,  
 $AB, BC, CA$  are equal to  $EB, BC, CE$  respectively,  
 the angle  $ABC$  is equal to the angle  $EBC$ . (I. Prop. 8.)

And because in the triangles  $ABF, EBF$ ,  
 $AB, BF$  are equal to  $EB, BF$  respectively,  
 and the angle  $ABF$  to the angle  $EBF$ ,  
 $AF$  is equal to  $EF$ . (I. Prop. 4.)

And because in the triangles  $ADF$ ,  $EDF$ ,  
 $AD$ ,  $DF$ ,  $FA$  are equal to  $ED$ ,  $DF$ ,  $FE$  respectively,  
the angle  $ADF$  is equal to the angle  $EDF$ : (I. Prop. 8.)  
that is,  $DF$  any straight line through  $D$  in the plane  $BCD$   
is at right angles to  $DA$ ;  
therefore  $DA$  is at right angles to the plane  $BDC$ .  
(Def. 5.)

Wherefore, *if one of three &c.*

### EXERCISES.

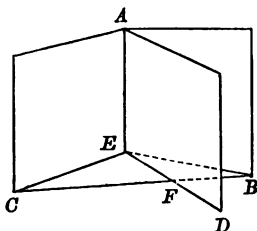
1. If from a point  $A$  without a plane  $BCD$ , there be drawn two straight lines  $AB$ ,  $AC$ , of which  $AC$  is at right angles to the plane, then the angle  $ABC$  is the inclination of  $AB$  to the plane  $BCD$ . (See Def. 8.)
2. Equal straight lines drawn from a given point to a given plane are equally inclined to the plane.
3. If two straight lines in one plane be equally inclined to another plane, they are equally inclined to the common section of these planes.
4. Through a given point draw a plane at right angles to a given straight line.

## PROPOSITION 5.

*If one of four concurrent straight lines be at right angles to the other three, those three lie in a plane.*

Let  $EA, EB, EC, ED$  be four concurrent straight lines such that the angles  $AEB, AEC, AED$  are right angles: it is required to prove that  $EB, EC, ED$  lie in a plane.

CONSTRUCTION. Draw any straight line  $BC$  cutting  $EB$  and  $EC$ , and let it cut the plane  $AED$  in  $F$ ; draw  $EF$ .



PROOF. Because  $EA$  is at right angles to  $EB$  and  $EC$ , it is at right angles to the plane  $BEC$ . (Prop. 4.)

And  $EF$  lies in this plane;

therefore the angle  $AEF$  is a right angle.

And the angle  $AED$  is a right angle, (Hypothesis) and the angles are in the same plane;

therefore  $EF$  and  $ED$  are in the same straight line, that is,  $EB, EC, ED$  lie in a plane.

Wherefore, *if one of four &c.*

COROLLARY. *Only one plane can be drawn at right angles to a given straight line through a given point on it.*

## EXERCISES.

1. If  $OP$  make equal angles with three coplanar straight lines  $OQ$ ,  $OR$ ,  $OS$ ,  $OP$  is at right angles to the plane  $QRS$ .
2. Prove that, if two spheres intersect, their curve of section is a circle.

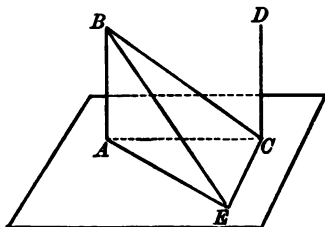
## PROPOSITION 6.

*If two straight lines be at right angles to the same plane, they are parallel to one another.*

Let  $AB$ ,  $CD$  be two straight lines at right angles to the plane  $ACE$ :

it is required to prove that  $AB$ ,  $CD$  are parallel to one another.

CONSTRUCTION. Draw  $AC$ , and in the plane  $ACE$  draw  $CE$  at right angles to  $AC$ . Take  $AB$ ,  $CE$  equal to one another and draw  $AE$ ,  $EB$ ,  $BC$ .



PROOF. Because  $AB$  is at right angles to the plane  $ACE$ , each of the angles  $BAC$ ,  $BAE$  is a right angle. (Def. 5.)

Because in the triangles  $ECA$ ,  $BAC$ ,

$EC$ ,  $CA$  are equal to  $BA$ ,  $AC$  respectively,

and the angle  $ECA$  to the angle  $BAC$ ,

therefore  $EA$  is equal to  $BC$ . (I. Prop. 4.)

And because in the triangles  $ECB$ ,  $BAE$ ,

$EC$ ,  $CB$ ,  $BE$  are equal to  $BA$ ,  $AE$ ,  $EB$  respectively,

the angle  $ECB$  is equal to the angle  $BAE$ , (I. Prop. 8.)

which has been proved to be a right angle.

And because  $CD$  is at right angles to the plane  $ACE$ ,

the angle  $DCE$  is a right angle;

therefore each of the angles  $ECD$ ,  $ECB$ ,  $ECA$  is a right angle;

therefore the lines  $CD$ ,  $CB$ ,  $CA$  lie in a plane. (Prop. 5.)

Also  $AB$ ,  $CB$ ,  $CA$  lie in a plane,

and the angles  $BAC$ ,  $ACD$  are right angles,

therefore  $AB$  is parallel to  $CD$ . (I. Prop. 28.)

Wherefore, if two straight lines &c.



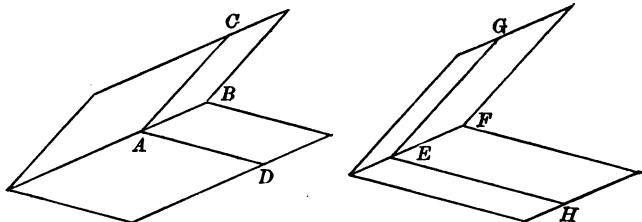
**EXERCISES.**

1. Only one straight line can be drawn from a given point at right angles to a given plane.
2. If each of two straight lines be at right angles to one of two parallel planes they are parallel to one another.

## PROPOSITION 7.

*If the dihedral angle between a pair of planes at a point be equal to the dihedral angle between another pair of planes at a point, the inclination of the first pair of planes at the first point is equal to the inclination of the second pair of planes at the second point.*

Let  $ABC$ ,  $ABD$  be one pair of planes, and  $EFG$ ,  $EFH$  another pair of planes such that the dihedral angle at the point  $A$  between the first pair of planes is equal to the dihedral angle at the point  $E$  between the second pair of planes; and let  $AC$ ,  $AD$  be drawn in the planes  $ABC$ ,  $ABD$  at right angles to  $AB$ ; and  $EG$ ,  $EH$  in the planes  $EFG$ ,  $EFH$  at right angles to  $EF$ :



it is required to prove that the plane angle  $CAD$  is equal to the plane angle  $GEH$ . (Def. 7.)

Because the dihedral angles at  $A$  and  $E$  are equal it is possible to shift the figure  $ABCD$ , so that the point  $A$  shall coincide with the point  $E$ , the plane  $ABC$  with the plane  $EFG$  and the plane  $ABD$  with the plane  $EFH$ .

If this be done,

because the plane  $ABC$  coincides with the plane  $EFG$ ; and the plane  $ABD$  with the plane  $EFH$ ,  $AB$  must coincide in direction with  $EF$ ,

and because the plane  $ABC$  coincides with the plane  $EFG$ , and  $AB$  coincides in direction with  $EF$ ,

and the angle  $BAC$  is equal to the angle  $FEG$ ,

each being a right angle,

$AC$  coincides in direction with  $EG$ ;

in a similar manner it can be proved that

$AD$  coincides in direction with  $EH$ :

therefore the plane angle  $CAD$  coincides with the plane angle  $GEH$  and is equal to it.

Wherefore, *if the dihedral angle &c.*

COROLLARY. If two adjacent dihedral angles between two planes at a point be equal, the inclination of the planes to one another at the point is a right angle.

It has been proved in Proposition 7 that, if the dihedral angles between two pairs of planes at two points be equal, the inclinations of the pairs of planes at the two points are equal.

By a method exactly similar to that used in Proposition 1 of Book VI. we can prove the following theorem, viz.

*The ratio of the dihedral angles between two pairs of planes at two points is equal to the ratio of the inclinations of the pairs of planes at the two points.*

On this account the inclination of two planes at a point is often used as a **measure of the dihedral angle** between the planes at the point.

The student must recollect that the dihedral angle between two planes is a quantity of an essentially different nature from the inclination of the planes.

#### EXERCISE.

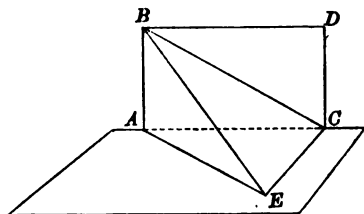
1. Draw a plane to bisect the dihedral angle between two given planes at a given point in their common section.

## PROPOSITION 8.

*If one of two parallel straight lines be at right angles to a plane, the other is also at right angles to the plane.*

Let the two straight lines  $AB$ ,  $CD$  be parallel, and let  $AB$  be at right angles to the plane  $ACE$ : it is required to prove that  $CD$  is at right angles to the plane  $ACE$ .

CONSTRUCTION. Draw  $AC$ , and in the plane  $ACE$  draw  $CE$  at right angles to  $AC$ ; make  $CE$  equal to  $AB$ , and draw  $AE$ ,  $EB$ ,  $BC$ .



PROOF. Because in the triangles  $BAC$ ,  $ECA$ ,  
 $BA$ ,  $AC$  are equal to  $EC$ ,  $CA$  respectively,  
 and the angle  $BAC$  to the angle  $ECA$ ,  
 $BC$  is equal to  $EA$ . (I. Prop. 4.)

And because in the triangles  $ECB$ ,  $BAE$ ,  
 $BC$ ,  $CE$ ,  $EB$  are equal to  $EA$ ,  $AB$ ,  $BE$  respectively,  
 the angle  $ECB$  is equal to the angle  $BAE$ , (I. Prop. 8.)  
 which is a right angle; (Hypothesis)

that is,  $EC$  is at right angles to  $CB$ ;

and it is at right angles to  $CA$ ; (Constr.)

therefore it is at right angles to  $CD$  a straight line in the plane  $ACB$ . (Prop. 4.)

And because  $CD$  is parallel to  $AB$ ,

and the angle  $BAC$  is a right angle, (Hypothesis)

therefore the angle  $DCA$  is a right angle. (I. Prop. 29.)

Therefore  $CD$  is at right angles to the two straight lines  $CA$ ,  $CE$ ,

and therefore to the plane  $ACE$ . (Prop. 4.)

Wherefore, if one of two &c.

## EXERCISES.

1. If perpendiculars be drawn from a point to a plane, and to a straight line in the plane, the straight line joining the feet of the perpendiculars is at right angles to the given straight line.
2.  $AB$  is at right angles to a plane at  $B$ : from  $B$ ,  $BD$  is drawn perpendicular to a straight line  $EF$  in the plane: prove that  $AD$  is at right angles to  $EF$ .
3.  $AB$ ,  $CD$  are two straight lines, of which  $AB$  lies in a plane to which  $CD$  is perpendicular. Prove that the perpendiculars to  $AB$  from the different points of  $CD$  all pass through a fixed point.
4. If  $D$  be any point on the straight line drawn through the orthocentre of a triangle  $ABC$  perpendicular to the plane of the triangle, then  $DA$  is at right angles to the straight line drawn through  $A$  parallel to  $BC$ .

## PROPOSITION 9.

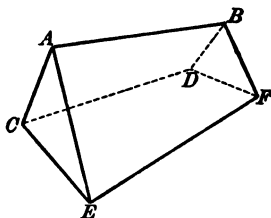
*Straight lines parallel to the same straight line are parallel to one another\*.*

Let  $AB$ ,  $CD$ ,  $EF$  be three given straight lines, such that  $AB$ ,  $EF$  are parallel, and  $CD$ ,  $EF$  are parallel :

it is required to prove that  $AB$ ,  $CD$  are parallel.

CONSTRUCTION. Take any point  $C$  in  $CD$  and draw the plane  $ACB$ ,

and let it meet the plane  $CEF$  in the straight line  $CG$ .



PROOF. Because the three planes  $AEFB$ ,  $ABC$ ,  $CEF$  meet in pairs in the lines  $CG$ ,  $EF$ ,  $AB$ ,

$CG$ ,  $EF$ ,  $AB$  meet in a point or are parallel in pairs.

(Prop. 3.)

They do not meet in a point, since the pairs  $AB$ ,  $EF$  are parallel,

therefore they are parallel in pairs ;

therefore  $CG$  is parallel to  $AB$  and to  $EF$ .

Now  $CD$  is parallel to  $EF$ ; (Hypothesis)

therefore  $CD$  is coincident with  $CG$ ;

therefore  $CD$  is parallel to  $AB$ .

Wherefore, *straight lines &c.*

\* The case when they are all in one plane has already been treated, Book I, Prop. 30.

## EXERCISES.

1. The straight lines joining the middle points of opposite edges of a tetrahedron meet in a point, and bisect one another.
2. If two of the straight lines joining the middle points of two pairs of opposite edges of a tetrahedron be at right angles, the remaining edges are of equal length.
3. If two opposite edges of a tetrahedron be equal, the section of the tetrahedron by a plane parallel to these edges is a parallelogram of constant perimeter.
4. If any two planes be cut by another plane, which is parallel to the line of intersection of the first two planes, their common sections with it are parallel.

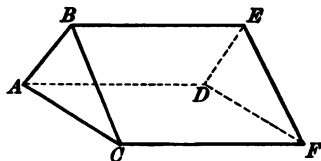
## PROPOSITION 10.

*The angles made by a pair of intersecting straight lines are equal to the angles made by any pair of intersecting straight lines parallel to them.*

Let the intersecting straight lines  $AB$ ,  $AC$  be parallel to the intersecting straight lines  $DE$ ,  $DF$  respectively: it is required to prove that the angles  $BAC$ ,  $EDF$  are equal.

CONSTRUCTION. Draw  $AD$ , and any two straight lines  $BE$ ,  $CF$ , in the planes  $ABED$ ,  $ACFD$  respectively, each parallel to  $AD$ .

Draw  $BC$ ,  $EF$ .



PROOF. Because  $ABED$  is a parallelogram,  
 $BE$  is equal to  $AD$ , and  $AB$  to  $DE$ ;  
 and because  $ACFD$  is a parallelogram,  
 $CF$  is equal to  $AD$ , and  $AC$  to  $DF$ ;  
 therefore  $CF$  is equal to  $BE$ .

Because  $BE$  and  $CF$  are each parallel to  $AD$ ,  
 they are parallel to each other; (Prop. 9.)

and it has been proved that they are equal;

therefore  $BC$ ,  $EF$  are equal and parallel. (I. Prop. 33.)

And because in the triangles  $BAC$ ,  $EDF$ ,  
 $BA$ ,  $AC$ ,  $CB$  are equal to  $ED$ ,  $DF$ ,  $FE$  respectively,  
 the angle  $BAC$  is equal to the angle  $EDF$ . (I. Prop. 8.)

Wherefore, *the angles &c.*

COROLLARY 1. The inclination of two planes is constant at all points of their common section.

COROLLARY 2. The dihedral angle between two planes is constant at all points of their common section. (See page 521.)



**DEFINITION.** *Two non-intersecting straight lines are said to be inclined to each other at the same angles as any two intersecting straight lines parallel to them.*

### EXERCISES.

1. If two planes which are not parallel be cut by two parallel planes, the lines of section of the first two planes by each of the others will contain equal angles.
2. Prove that in the figure of Proposition 8,  $DE$  is at right angles to  $AC$ .
3. If in a trihedral angle  $A(BCD)$  the plane angles  $BAC$ ,  $CAD$  be equal, the inclinations of the planes  $BAC$ ,  $BAD$  and  $CAD$ ,  $BAD$  are equal.
4. If in a trihedral angle  $A(BCD)$  the plane angle  $BAC$  be greater than the plane angle  $CAD$ , the inclination of the planes  $BAC$ ,  $BAD$  is greater than the inclination of the planes  $CAD$ ,  $BAD$ .
5. Prove that every straight line which is equally inclined to two given planes is parallel to one or other of the two planes that bisect the dihedral angle between the given planes.
6. Draw through the vertex of a trihedral angle a straight line to be equally inclined to the three faces of the trihedral angle.
7. A pyramid stands on a regular polygon as base, and has equal isosceles triangles for its faces; prove that the plane bisecting the angle between the base and any face divides the perpendicular from the vertex upon the base into segments which are in the ratio of the area of the base to the sum of the areas of the faces.

## PROPOSITION 11.

*To draw a straight line at right angles to a given plane from a given point without it.*

Let  $ABC$  be a given plane and  $D$  a given point without it:

it is required to draw from  $D$  a straight line at right angles to the plane  $ABC$ .

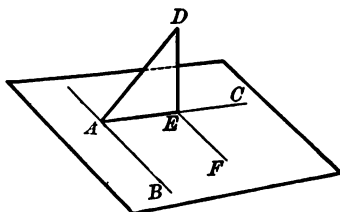
CONSTRUCTION. Draw in the plane  $ABC$  any straight line  $AB$ .

In the plane  $DAB$  draw  $DA$  at right angles to  $AB$ :  
if  $DA$  be at right angles to the plane  $ABC$ , what is required is done. If it be not, in the plane  $CAB$  draw  $AC$  at right angles to  $AB$ ;

and in the plane  $CAD$  draw  $DE$  at right angles to  $AC$ :

$DE$  is the line required.

Draw  $EF$  parallel to  $AB$ .



PROOF. Because each of the angles  $BAD$ ,  $BAC$  is a right angle, (Constr.)

$BA$  is at right angles to the plane  $DAC$ ;

and because  $EF$  is parallel to  $AB$ ,

$EF$  is at right angles to the plane  $DAC$ ; (Prop. 8.)

therefore the angle  $DEF$  is a right angle;

and the angle  $DEA$  is a right angle; (Constr.)

therefore  $DE$  is at right angles to the plane  $ABC$ , which passes through  $EA$  and  $EF$ . (Prop. 4.)

Wherefore,  $DE$  has been drawn at right angles to the given plane  $ABC$  from the given point  $D$  without it.

## EXERCISES.

1. If the sides  $AD$ ,  $BD$ ,  $CD$  of the tetrahedron  $ABCD$ , be equal, each of them is greater than the radius of the circle circumscribing the triangle  $ABC$ .

2. In a regular tetrahedron twice the square on an edge is equal to three times the square on the perpendicular from a vertex on the opposite face.

3. From a point  $E$  draw  $EC$ ,  $ED$  perpendicular to two planes  $CAB$ ,  $DAB$  which intersect in  $AB$ , and from  $D$  draw  $DF$  perpendicular to the plane  $CAB$  meeting it at  $F$ : shew that the straight line  $CF$  will cut  $AB$  at right angles.

4. The areas of the four faces of a pyramid, whose base is a triangle, are equal: prove that the perpendiculars from the angles on the opposite faces are equal. [The volume of the pyramid is *not* to be employed in the proof.]

5. On a given triangle as base a tetrahedron is described having all the angles at its vertex right angles. Shew that the perpendicular from the vertex meets the base where the perpendiculars from the angles on the opposite sides intersect.

6. Straight lines  $AP$ ,  $BP$  are drawn from two given points  $A$ ,  $B$  without a given plane to a point  $P$  in the plane; find when their sum is the least possible.

7. Prove that the locus of the feet of the perpendiculars drawn from a fixed point on planes passing through a fixed point is a sphere.

8. Prove that the locus of the feet of the perpendiculars drawn from a fixed point upon planes passing through a fixed straight line is a circle.

## PROPOSITION 12.

*To draw a straight line at right angles to a given plane from a given point in it.*

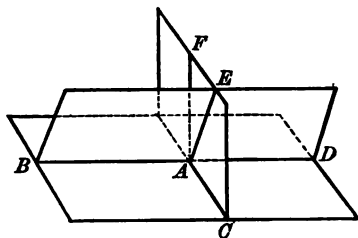
Let  $A$  be a given point in the plane  $ABC$  :  
it is required to draw a straight line from  $A$  at right angles  
to the plane  $ABC$ .

CONSTRUCTION. In the plane  $ABC$  draw any straight  
line  $BD$ , and  $AC$  at right angles to it.

Through  $BD$  draw any plane  $BAED$ ,  
and in it draw  $AE$  at right angles to  $BD$ .

If the angle  $CAE$  be a right angle, what is required is  
done : but if not, draw the plane  $CAE$  and in it draw  $AF$   
at right angles to  $AC$  :

$AF$  is the line required.



PROOF. Because  $BA$  is at right angles to  $AC$  and  $AE$ ,  
it is at right angles to the plane  $ACE$  and therefore to  
every line in that plane ; (Prop. 4.)

therefore  $BAF$  is a right angle.

And  $FAC$  is a right angle ; (Constr.)

therefore  $AF$  is at right angles to the plane  $BAC$ .

(Prop. 4.)

Wherefore,  $AF$  has been drawn at right angles to the  
given plane  $ABC$  from the given point  $A$  in it.

## EXERCISES.

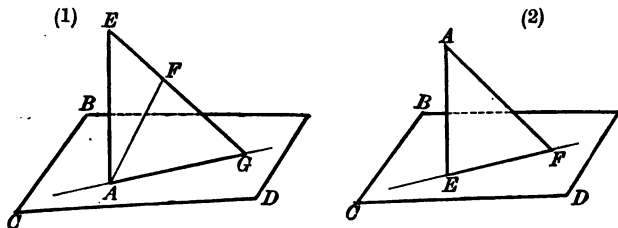
1. On a given equilateral triangle as base construct a regular tetrahedron.
2. Draw through the vertex of a trihedral angle a straight line to make equal angles with the three edges of the trihedral angle.  
How many solutions are there?
3. On a given triangle as base construct a tetrahedron such that the other three edges are equal to three given straight lines.

## PROPOSITION 13.

*Only one straight line can be drawn through a given point at right angles to a given plane.*

Let  $A$  be a given point and  $BCD$  a given plane, and let  $AE$  be a straight line at right angles to the plane  $BCD$ : it is required to prove that no other straight line through  $A$  can be at right angles to the plane  $BCD$ .

CONSTRUCTION. Draw  $AF$  any other straight line from  $A$ , and let the plane  $EAFF$  be drawn :  
in fig. 1, in which  $A$  is in the plane  $BCD$ , let the plane  $EAFF$  cut the plane  $BCD$  in  $AG$ ;  
and in fig. 2, in which  $A$  is without the plane  $BCD$ , let the plane  $EAFF$  cut the plane  $BCD$  in  $EF$ .



PROOF. In fig. 1, because  $AE$  is at right angles to the plane  $BCD$ , the angle  $EAG$  is a right angle;

therefore the angle  $FAG$  is not a right angle;  
therefore, in fig. 1,  $AF$  is not at right angles to the plane  $BCD$ .

In fig. 2, because  $AE$  is at right angles to the plane  $BCD$ , the angle  $AEF$  is a right angle;  
therefore the angle  $AFF$  is not a right angle; (I. Prop. 17.)  
therefore, in fig. 2,  $AF$  is not at right angles to the plane  $BCD$ .

Wherefore, *only one straight line &c.*

**DEFINITION.** *If from a given point a straight line be drawn at right angles to a plane the point, where the straight line meets the plane, is called the projection of the given point on the plane.*

*If from a given point a straight line be drawn at right angles to a given straight line the point of intersection of the straight lines is called the projection of the given point on the given straight line.*

*The straight line joining the projections of two given points is called the projection of the straight line joining the points.*

#### EXERCISES.

1. The projections on the same straight line of two equal and parallel straight lines are equal.
2. The projections of a straight line on two parallel straight lines are equal.
3. The projections on the same plane of two equal and parallel straight lines are equal and parallel.
4. The square of the distance between two points is equal to the sum of the squares of its projections on three straight lines at right angles to each other.

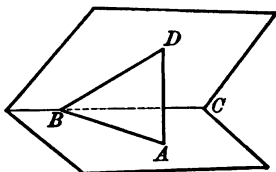
## PROPOSITION 14.

*Two intersecting planes cannot both be at right angles to the same straight line.*

Let the two planes  $ABC$ ,  $DBC$  intersect in the straight line  $BC$ , and let the straight line  $AD$  be at right angles to the plane  $ABC$ :

it is required to prove that  $AD$  is not at right angles to the plane  $BCD$ .

CONSTRUCTION. Take any point  $B$  in  $BC$ , and draw  $AB$ ,  $BD$ .



PROOF. Because  $AD$  is at right angles to the plane  $ABC$ ,  
the angle  $BAD$  is a right angle;  
therefore the angle  $ADB$  is less than a right angle;

(I. Prop. 17.)  
therefore  $AD$  is not at right angles to the plane  $BCD$ .

Wherefore, *two intersecting planes &c.*

COROLLARY. *Planes, which are at right angles to the same straight line, are parallel to one another.*



**EXERCISES.**

1. Through a given point only one plane can be drawn parallel to a given plane.

2. In general, through two given points only one plane can be drawn parallel to a given straight line.

What are the cases of exception?

3. If a straight line be at right angles to one of two parallel planes it is at right angles to the other.

4. The projections of a straight line on two parallel planes are equal and parallel.

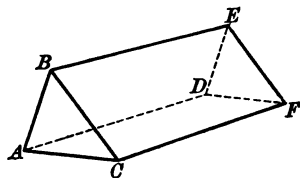
## PROPOSITION 15.

*The plane through a pair of intersecting straight lines is parallel to the plane through any pair of intersecting straight lines parallel to them.*

Let the intersecting straight lines  $AB, AC$  be parallel to the intersecting straight lines  $DE, DF$  respectively: it is required to prove that the planes  $ABC, DEF$  are parallel.

CONSTRUCTION. Draw  $AD$ , and in the planes  $ABED, ACFD$  draw two straight lines  $BE, CF$  parallel to  $AD$ .

Draw  $BC$  and  $EF$ .



PROOF. Because each of the lines  $BE, CF$  is parallel to  $AD$ , they are parallel to each other and therefore lie in a plane. (Prop. 9.)

$ABC, DEF, ABED$  are three planes and  $AB, DE$  two of their intersections in pairs are parallel; therefore if  $ABC, DEF$  intersected, their intersection would be parallel to  $AB$  and  $DE$ . (Prop. 3.)

$ABC, DEF, ACFD$  are three planes, and  $AC, DF$  two of their intersections in pairs are parallel; therefore if  $ABC, DEF$  intersected, their intersection would be parallel to  $AC$  and  $DF$ . (Prop. 3.)

The intersection of the planes  $ABC, DEF$  cannot be parallel both to  $AB$  and  $DE$  and also to  $AC$  and  $DF$ ;

therefore the planes  $ABC, DEF$  do not intersect, that is, they are parallel.

Wherefore, the plane &c.

## EXERCISES.

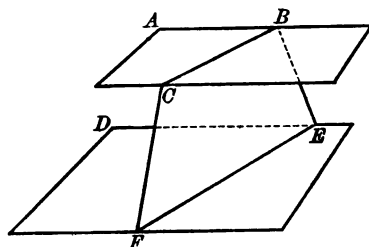
1. Draw a plane through one straight line parallel to another straight line.
2. Draw two parallel planes, one through each of two non-intersecting straight lines.
3. A point  $O$  is the middle point of each of the straight lines  $AA'$ ,  $BB'$ ,  $CC'$ ; prove that the planes  $ABC$ ,  $A'B'C'$  are parallel.
4. The point  $O$  divides the straight lines  $AA'$ ,  $BB'$ ,  $CC'$  proportionally: prove that the perpendiculars from  $O$  on the two planes  $ABC$ ,  $A'B'C'$  are in the same straight line.
5. Draw a plane through a given point parallel to a given plane.
6. Given the lengths and positions of two straight lines which do not meet when produced and are not parallel, form a parallelepiped of which these two lines shall be two of the edges.
7. Two intersecting lines  $OA$ ,  $OB$  are drawn parallel to two non-intersecting lines  $P$ ,  $Q$  respectively; prove that the shortest distance between the lines  $P$ ,  $Q$  can be found by drawing a line perpendicular to the plane  $OAB$  at the point of intersection of the projections of  $P$  and  $Q$  on the plane  $OAB$ .

## PROPOSITION 16.

*The sections of two parallel planes made by a third plane are parallel straight lines.*

Let  $ABC$ ,  $DEF$  be two parallel planes and let them be cut by the plane  $BCFE$  in the straight lines  $BC$ ,  $EF$  respectively :

it is required to prove that  $BC$ ,  $EF$  are parallel.



**PROOF.** Every point in  $BC$  is in the plane  $ABC$ ;  
and every point in  $EF$  is in the plane  $DEF$ ;  
therefore if  $BC$ ,  $EF$  met, the planes  $ABC$ ,  $DEF$  would meet.

But the planes  $ABC$ ,  $DEF$  are parallel,  
therefore  $BC$ ,  $EF$  do not meet;  
and they are in the same plane  $BCFE$ ;  
therefore they are parallel.

Wherefore, *the sections &c.*

## EXERCISES.

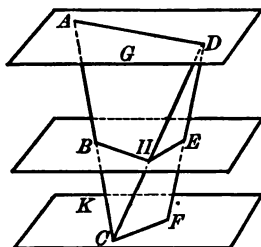
1. Polygons formed by cutting the faces of a polyhedral angle by parallel planes are similar to one another.
2. Polygons formed by cutting those faces of a prism, which are parallel to the same straight line, by parallel planes are equal.
3. Planes parallel to the same plane are parallel to one another.
4.  $ABCD$  is a rectangle: through three of the angular points  $A$ ,  $B$ ,  $C$  perpendiculars are drawn to the plane of the rectangle, and through the fourth point  $D$  any plane is drawn cutting these perpendiculars in points  $E$ ,  $F$ ,  $G$ ; prove that  $DEFG$  is always a parallelogram and may be a rhombus.
5. Shew how to draw a plane so that its section of a tetrahedron may be a parallelogram.
6. If a tetrahedron have two opposite edges at right angles, it is possible to draw a plane so that its intersection with the faces is a square.
7. A tetrahedron is cut by a plane so that the section shall be a rhombus; prove that the side of the rhombus is half the harmonic mean between a pair of opposite edges.

## PROPOSITION 17.

*If two straight lines be cut by parallel planes, they are cut proportionally.*

Let the two straight lines  $ABC$ ,  $DEF$  be cut by the three parallel planes  $ADG$ ,  $BEH$ ,  $CFK$ :  
it is required to prove that  $AB$  is to  $BC$  as  $DE$  to  $EF$ .

CONSTRUCTION. Draw  $AD$ ,  $DC$ ,  $CF$ . Let  $DC$  cut the plane  $BEH$  in  $H$  and draw  $BH$ ,  $HE$ .



PROOF. Because the plane  $ACD$  cuts the parallel planes  $ADG$ ,  $BEH$  in  $AD$  and  $BH$ ,

$AD$  is parallel to  $BH$ ; (Prop. 16.)

therefore  $AB$  is to  $BC$  as  $DH$  to  $HC$ . (VI. Prop. 2, Part 1.)

And because the plane  $DCF$  cuts the parallel planes  $BEH$ ,  $CFK$  in  $HE$  and  $CF$ ,

$HE$  is parallel to  $CF$ ; (Prop. 16.)

therefore  $DH$  is to  $HC$  as  $DE$  to  $EF$ ; (VI. Prop. 2, Part 1.)

therefore  $AB$  is to  $BC$  as  $DE$  to  $EF$ . (V. Prop. 5.)

Wherefore, if two straight lines &c.

## EXERCISES.

1. Three straight lines which do not all lie in one plane, are cut proportionally by three planes, two of which are parallel: shew that the third will be parallel to the other two, if its intersections with the three straight lines are not all in the same straight line.

2. Prove that a pyramid can be constructed so that its faces shall pass through any two similar polygons whose corresponding sides are parallel.

3. From the extremities of the two parallel straight lines  $AB$ ,  $CD$  parallel straight lines  $Aa$ ,  $Bb$ ,  $Cc$ ,  $Dd$  are drawn meeting a plane at  $a$ ,  $b$ ,  $c$ ,  $d$ : shew that  $AB$  is to  $CD$  as  $ab$  to  $cd$ .

4. If the intersections of four parallel straight lines with any plane form the angular points of a parallelogram, their intersections with every plane that meets them form the angular points of a parallelogram.

5. Draw a plane parallel to each of two given non-intersecting straight lines and equidistant from them.

6. The straight lines  $AD$ ,  $BE$ ,  $CF$ , joining points on two given non-intersecting straight lines  $ABC$ ,  $DEF$ , are divided at the points  $P$ ,  $Q$ ,  $R$  in the same ratio.

Prove that the plane  $PQR$  is parallel to the two lines  $ABC$  and  $DEF$ .

7. Draw a straight line  $PQR$  to cut three given non-intersecting straight lines  $AB$ ,  $CD$ ,  $EF$  in  $P$ ,  $Q$ ,  $R$  respectively so that  $PQ$  may be to  $QR$  in a given ratio.

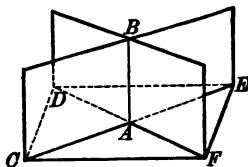
## PROPOSITION 18.

*If a straight line be at right angles to a plane, any plane through the line is at right angles to the plane.*

Let the straight line  $AB$  be at right angles to the plane  $CDEF$ :

it is required to prove that any plane through  $AB$  is at right angles to the plane  $CDEF$ .

CONSTRUCTION. Through  $AB$  draw any plane  $CBE$  cutting the plane  $CDEF$  in the straight line  $CAE$ . Through  $A$  draw  $DAF$  at right angles to  $CAE$ .



PROOF. Because  $AB$  is at right angles to the plane  $CDEF$ , the angle  $BAC$  is a right angle; and because  $DAF$  was drawn at right angles to  $CA$ , the angles  $CAD$ ,  $CAF$  are right angles.

Therefore the angles  $BAD$ ,  $BAF$  are the inclinations of the plane  $CBEA$  to the plane  $CDAF$  on either side of it, at the point  $A$ , (Def. 7.)

and they are equal because  $AB$  is at right angles to the plane  $CDEF$ .

Therefore the dihedral angles, which the plane  $CBEA$  makes with the plane  $CDAF$  on either side of it at the point  $A$  are equal, and the plane  $CBEA$  is at right angles to the plane  $CDEF$  at the point  $A$  and therefore at all points of their common section. (See p. 521.)

Wherefore, *if a straight line &c.*



## EXERCISES.

1. Through a given straight line draw a plane perpendicular to a given plane.

Is the solution always unique?

2. Perpendiculars  $AE$ ,  $BF$  are drawn to a plane from two points  $A$ ,  $B$  above it; a plane is drawn through  $A$  perpendicular to  $AB$ : shew that its line of intersection with the given plane is perpendicular to  $EF$ .

3. Through a given point draw a plane at right angles to each of two given planes.

4. The straight line, which is the shortest distance between two non-intersecting straight lines, is at right angles to each of them.

5. Draw a straight line to cut at right angles each of two given non-intersecting straight lines.

Is the solution unique?

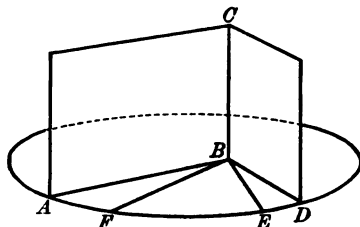
## PROPOSITION 19.

*If two intersecting planes be at right angles to the same plane, their common section is at right angles to it.*

Let each of the given planes  $ABC$ ,  $DBC$  be at right angles to the plane  $ABD$ , and let them intersect in the straight line  $BC$ :

it is required to prove that  $BC$  is at right angles to the plane  $ABD$ .

CONSTRUCTION. In the plane  $ABD$  draw  $BE$ ,  $BF$  making the angles  $ABE$ ,  $DBF$  right angles.



PROOF. Because  $BE$  is drawn in the plane  $ABD$  at right angles to  $AB$ , the intersection of the planes  $ABD$ ,  $ABC$ ,

$BE$  is at right angles to the plane  $ABC$ ;

(Prop. 7, Coroll.).

therefore the angle  $CBE$  is a right angle.

Similarly it can be proved that

the angle  $CBF$  is a right angle;

therefore  $BC$  is at right angles to the plane through  $BE$ ,  $BF$ , that is to the plane  $ABD$ .

(Prop. 4.)

Wherefore, if two intersecting planes &c.

## EXERCISES.

1. If three planes be at right angles to each other in pairs at a point, their common sections are at right angles to each other in pairs.

2. From a point  $A$  in one of two planes are drawn  $AB$  at right angles to the first plane, and  $AC$  perpendicular to the second plane, and meeting the second plane at  $B, C$ : shew that  $BC$  is at right angles to the line of intersection of the two planes.

3. Two straight lines  $AB, DC$  are such that if  $BC, AD$  be joined, each of the angles at  $A, B, C, D$  is a right angle. Prove that  $AB, DC$  are parallel.

4.  $OA, OB, OC$  are three straight lines, not in the same plane: planes through  $OB, OC$  at right angles to  $OBC$  intersect in  $Oa$ : planes through  $OC, OA$  at right angles to  $OCA$  intersect in  $Ob$ , and planes through  $OA, OB$  at right angles to  $OAB$  intersect in  $Oc$ : prove that  $OA, OB, OC$  are at right angles to the planes  $Obc, Oca, Oab$  respectively.

## PROPOSITION 20.

*The sum of any two plane angles of a trihedral angle is greater than the third angle.*

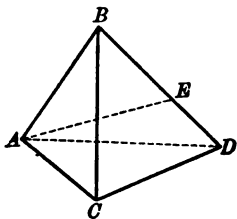
Let the trihedral angle at  $A$  be contained by the three plane angles  $BAC$ ,  $CAD$ ,  $DAB$ :

it is required to prove that the sum of any two of these angles is greater than the third angle.

CONSTRUCTION. Let the angle  $BAD$  be not less than either of the angles  $BAC$ ,  $CAD$ .

In the plane  $BAD$  draw the straight line  $AE$  making the angle  $BAE$  equal to the angle  $BAC$ , and draw any straight line  $BED$ , meeting  $AB$ ,  $AD$  in  $B$ ,  $D$ .

Take  $AC$  equal to  $AE$ , and draw  $BC$ ,  $CD$ .



PROOF. Because in the triangles  $BAC$ ,  $BAE$ ,  
 $BA$ ,  $AC$  are equal to  $BA$ ,  $AE$  respectively,  
 and the angle  $BAC$  to the angle  $BAE$ , (Constr.)  
 $BC$  is equal to  $BE$ . (I. Prop. 4.)

And because  $BC$ ,  $CD$  are greater than  $BD$ ,  
 and  $BC$  is equal to  $BE$ ,  
 therefore  $CD$  is greater than  $ED$ .

And because in the two triangles  $CAD$ ,  $EAD$ ,  
 $CA$ ,  $AD$  are equal to  $EA$ ,  $AD$  respectively,  
 and  $CD$  is greater than  $ED$ ,  
 therefore the angle  $CAD$  is greater than the angle  $EAD$ .  
 (I. Prop. 25.)

Therefore the angles  $BAC$ ,  $CAD$  are greater than the angles  
 $BAE$ ,  $EAD$ ,  
 that is, than the angle  $BAD$ .

Wherefore, the sum &c.

**COROLLARY.** *If a solid angle be contained by more than three plane angles, any one is less than the sum of the remaining angles.*

## EXERCISES.

1. If the opposite edges of a tetrahedron be equal two and two, then the faces are acute-angled triangles.
2. If the faces of a tetrahedron be equal triangles, each is an acute-angled triangle.
3. A tetrahedron can be constructed whose faces are equal to any four equal acute-angled triangles.
4. If a point  $P$  be taken within a tetrahedron  $ABCD$ , the sum of the angles  $ABC$ ,  $ABD$  is greater than the sum of the angles  $PBC$ ,  $PBD$ .
5. If a point  $P$  be taken within a tetrahedron  $ABCD$ , the sum of the angles  $BPC$ ,  $CPA$ ,  $APB$  is greater than the sum of the angles  $BDC$ ,  $CDA$ ,  $ADB$ .
6. If four straight lines intersect in a point, and one fall within the solid angle formed by the other three, the sum of the angles which it makes with them is less than the sum but greater than half the sum of the angles which they make with one another.
7. Within the area of a given triangle is inscribed another triangle: shew that the sum of the angles subtended by the sides of the interior triangle at any point not in the plane of the triangles is less than the sum of the angles subtended at the same point by the sides of the exterior triangle.
8. At each of the corners  $B$ ,  $C$ ,  $D$  of a tetrahedron  $ABCD$  the sum of the three angles forming the solid angle is equal to two right angles. Prove that the sum of the three angles at  $A$  is also equal to two right angles, and that the four triangles  $ABC$ ,  $ACD$ ,  $ADB$ ,  $BCD$  are equal in all respects.

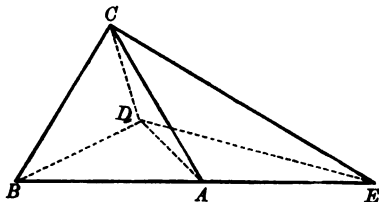
## PROPOSITION 21. PART 1.

*The sum of the plane angles of a trihedral angle is less than four right angles.*

Let the trihedral angle at  $A$  be contained by the angles  $BAC$ ,  $CAD$ ,  $DAB$ :

it is required to prove that the sum of these angles is less than four right angles.

CONSTRUCTION. Produce the planes  $BAC$ ,  $BAD$  through the lines  $AC$ ,  $AD$ ; the produced parts,  $CAE$ ,  $DAE$ , will meet in  $AE$ , which is in the same straight line with  $BA$ .



PROOF. Because a solid angle is formed at  $A$  by the planes  $DAC$ ,  $CAE$ ,  $EAD$ , the angle  $CAD$  is less than the sum of the angles  $CAE$ ,  $EAD$ ; (Prop. 20.) therefore the sum of the angles  $BAC$ ,  $CAD$ ,  $DAB$  is less than the sum of the angles  $BAC$ ,  $CAE$ ,  $EAD$ ,  $DAB$ , that is, less than four right angles.

(I. Prop. 13.)

Wherefore, *the sum &c.*

## EXERCISES.

1. The three edges  $DA$ ,  $DB$ ,  $DC$  of a tetrahedron  $ABCD$  are equal.  $DE$  is drawn at right angles to the plane  $ABC$  and  $EA$ ,  $EB$ ,  $EC$  are drawn. Prove that the angle  $AEB$  is greater than the angle  $ADB$ , and hence shew that the three plane angles of a trihedral angle are less than four right angles.

2. Use the method of proof adopted in Prop. 21, Part 1, to prove that the sum of the plane angles of any convex polyhedral angle is less than four right angles.

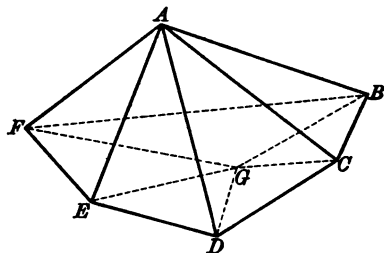
## PROPOSITION 21. PART 2.

*The sum of the plane angles of any convex polyhedral angle is less than four right angles.*

Let the convex polyhedral angle at  $A$  be contained by a number of plane angles  $BAC, CAD, DAE, EAF, FAB$ : it is required to prove that the sum of these angles is less than four right angles.

CONSTRUCTION. Draw any plane cutting the faces of the polyhedral angle at  $A$  in the convex polygon  $BCDEF$ .

Within the polygon take any point  $G$  and draw  $GB, GC, GD, GE, GF$ .



PROOF. Because the trihedral angle at  $B$  is contained by the angles  $ABF, ABC$  and  $FBC$ ,

the sum of the angles  $ABF, ABC$  is greater than  $FBC$ ,  
that is, than the sum of the angles  $GBF, GBC$ ;  
and similarly for the angles at  $C, D, E$  and  $F$ .

Therefore the sum of all the angles at  $B, C, D, E, F$  in the triangles  $ABC, ACD, ADE, AEF, AFB$  is greater than the sum of all the angles at  $B, C, D, E, F$  in the triangles  $GBC, GCD, GDE, GEF, GFB$ .

And the sum of all the angles of the first set of triangles is equal to the sum of all the angles of the second set of triangles which are of the same number.

Therefore the sum of all the angles at  $A$  is less than the sum of all the angles at  $G$ ,

that is, less than four right angles.

Wherefore, the sum of &c.



## EXERCISES.

1. If  $V$  and  $P$  be the vertices of two pyramids standing on the same convex polygon as base and if  $P$  be within the pyramid of which  $V$  is the vertex, the sum of the plane angles at  $P$  is greater than the sum of the plane angles at  $V$ .

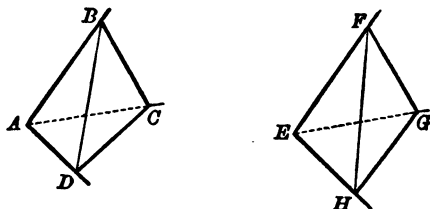
2. Within the area of any convex polygon is inscribed another convex polygon, shew that the sum of the angles subtended by the sides of the interior polygon at any point not in the plane of the polygon is less than the sum of the angles subtended at the same point by the sides of the exterior polygon.

## ADDITIONAL PROPOSITION 1.

*If two trihedral angles have their plane angles equal in pairs, the inclinations of corresponding pairs of faces are equal.*

Let  $A(BCD)$ ,  $E(FGH)$  be two trihedral angles whose plane angles are equal in pairs  $BAC$ ,  $CAD$ ,  $DAB$  to  $FEG$ ,  $GEH$ ,  $HEF$  respectively.

In the corresponding edges  $AC$ ,  $EG$ , take  $AC$  equal to  $EG$ ; and in the planes  $CAB$ ,  $CAD$  draw  $CB$ ,  $CD$  at right angles to  $AC$ , and in the planes  $GEF$ ,  $GEH$  draw  $GF$ ,  $GH$  at right angles to  $EG$ . Draw  $BD$ ,  $FH$ .



Because in the triangles  $BAC$ ,  $FEG$ ,  
the angles  $BAC$ ,  $ACB$  are equal to the angles  $FEG$ ,  $EGF$   
and  $AC$  is equal to  $EG$ ,  
therefore  $AB$  is equal to  $EF$ , and  $BC$  to  $FG$ . (I. Prop. 26, Part 1.)  
Similarly it may be proved that  
 $AD$  is equal to  $EH$  and  $DC$  to  $HG$ .

Because in the triangles  $BAD$ ,  $FEH$ ,  
 $BA$ ,  $AD$  are equal to  $FE$ ,  $EH$  and the angle  $BAD$  to the angle  $FEH$ ,  
therefore  $BD$  is equal to  $FH$ . (I. Prop. 4.)

And because in the triangles  $BCD$ ,  $FGH$ ,  
 $BD$ ,  $DC$ ,  $CB$  are equal to  $FH$ ,  $HG$ ,  $GF$ ,  
therefore the angle  $BCD$  is equal to the angle  $FGH$ . (I. Prop. 8.)

Therefore the inclination of the planes  $BAC$ ,  $DAC$  is equal to the inclination of the planes  $FEG$ ,  $HEG$ .

Similarly it can be proved that the inclinations of the other pairs of corresponding faces are equal.

The converse of this proposition is true, viz.

*If two trihedral angles have the inclinations of their faces equal in pairs, their corresponding plane angles are equal in pairs.*

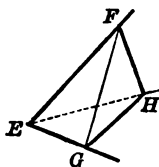
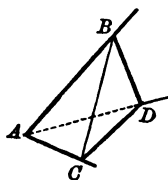
ADDITIONAL PROPOSITION 2.

*If two trihedral angles have two pairs of plane angles equal and the inclinations of the planes of those angles equal, the remaining pair of plane angles are equal.*

Let  $A(BCD)$ ,  $E(FGH)$  be two trihedral angles, in which the plane angles  $BAC$ ,  $CAD$  are equal to the plane angles  $FEG$ ,  $GEH$  respectively, and the inclination of the planes  $BAC$ ,  $CAD$  is equal to the inclination of the planes  $FEG$ ,  $GEH$ .

In the corresponding edges  $AC$ ,  $EG$ , take  $AC$  equal to  $EG$ ;  
and in the planes  $CAB$ ,  $CAD$  draw  $CB$ ,  $CD$  at right angles to  $CA$ ,  
and in the planes  $GEF$ ,  $GEH$  draw  $GF$ ,  $GH$  at right angles to  $GE$ .

Draw  $BD$ ,  $FH$ .



Because the inclinations of the planes at  $C$  and  $G$  are equal,  
and  $CB$ ,  $CD$  are at right angles to  $CA$ ,  
and  $GF$ ,  $GH$  are at right angles to  $GE$ ,  
the angle  $BCD$  is equal to the angle  $FGH$ .

Because in the triangles  $BAC$ ,  $FEG$ ,  
the angles  $BAC$ ,  $ACB$  are equal to the angles  $FEG$ ,  $EGF$ ,  
and  $AC$  is equal to  $EG$ ,  
therefore  $AB$  is equal to  $EF$ , and  $BC$  to  $FG$ . (I. Prop. 26, Part 1.)  
Similarly it can be proved that

$AD$  is equal to  $EH$  and  $DC$  to  $HG$ .

Because in the triangles  $BCD$ ,  $FGH$ ,  
 $BC$ ,  $CD$  are equal to  $FG$ ,  $GH$ ,  
and the angle  $BCD$  to the angle  $FGH$ ,  
therefore  $BD$  is equal to  $FH$ . (I. Prop. 4.)

Because in the triangles  $BAD$ ,  $FEH$ ,  
 $BA$ ,  $AD$ ,  $DB$  are equal to  $FE$ ,  $EH$ ,  $HF$  respectively,  
therefore the angle  $BAD$  is equal to the angle  $FEH$ . (I. Prop. 8.)

**COROLLARY.** *The other pairs of corresponding inclinations are also equal.*

## ADDITIONAL PROPOSITION 3.

*If two trihedral angles have two plane angles equal to two plane angles, and the inclinations of the faces meeting in the edges opposite one pair of equal plane angles equal, the inclinations of the faces meeting in the edges opposite to the other pair are either equal or supplementary.*

Let  $A(BCD)$ ,  $E(FGH)$  be two trihedral angles in which the plane angles  $BAC$ ,  $CAD$  are equal to the plane angles  $FEG$ ,  $GEH$  respectively, and the inclination of the planes  $BAD$ ,  $CAD$  equal to the inclination of the planes  $FEH$ ,  $GEH$ .

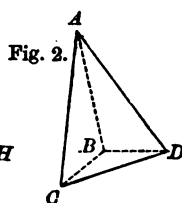
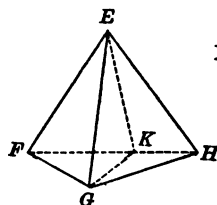
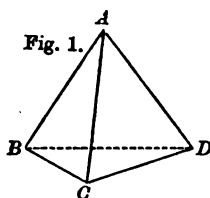
Of the two plane angles  $BAD$ ,  $FEH$ , let  $BAD$  be not greater than  $FEH$ .

Because the inclinations of the pairs of planes  $BAD$ ,  $CAD$ , and  $FEH$ ,  $GEH$  are equal,

the dihedral angles between the pairs of planes at the points  $A$  and  $E$  are equal;

therefore it is possible to shift the figure  $ABCD$  so that

$A$  coincides with  $E$ ,  
and  $AD$  coincides in direction with  $EH$ ,  
and the plane  $BAD$  with the plane  $FEH$ ,  
and the plane  $CAD$  with the plane  $GEH$ .



If this be done,

because  $A$  coincides with  $E$ ,  
and the line  $AD$  coincides in direction with  $EH$ ,  
and the angle  $CAD$  is equal to the angle  $GEH$ ,  
 $AC$  coincides in direction with  $EG$ .

Because the angle  $DAB$  is not greater than the angle  $HEF$ ,  
the line  $AB$  must coincide in direction either (1) with the line  $EF$  or  
(2) with some straight line  $EK$  in the angle  $FEH$ .

(Fig. 1.) If  $AB$  coincide in direction with  $EF$ ,  
the plane  $BAC$  coincides with the plane  $FEG$ ,  
and the inclinations of the pairs of planes  $BAC$ ,  $BAD$ , and  $FEG$ ,  
 $FEH$  are equal.

(Fig. 2.) If  $AB$  coincide in direction with  $EK$ ,  
because the angle  $BAC$  is equal to the angle  $KEG$ ,  
the angle  $FEG$  is equal to the angle  $KEG$ ,  
therefore the inclinations of the pairs of planes  $FEG$ ,  $FEK$ , and  
 $KEG$ ,  $KEF$  are equal (Ex. 5, page 527),  
and therefore the inclinations of the pairs of planes  $FEG$ ,  $FEH$ , and  
 $KEG$ ,  $KEH$  are supplementary;  
that is, the inclinations of the pairs of planes  $BAC$ ,  $BAD$ , and  $FEG$ ,  
 $FEH$  are supplementary.

## ADDITIONAL PROPOSITION 4.

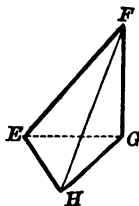
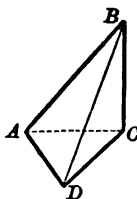
[Euclid, Elem. xi. Prop. 26.]

*To construct a trihedral angle equal to a given trihedral angle.*

Let  $A(BCD)$  be a given trihedral angle.

Take any point  $C$  in one of the edges  $AC$  and through  $C$  draw a plane  $BCD$  at right angles to  $AC$ , cutting the faces of the trihedral angle at  $A$  in the triangle  $BCD$ . Take any straight line  $EG$  equal to  $AC$ , and through  $G$  draw the plane  $FGH$  at right angles to  $GE$ . Draw in the plane  $FGH$  from  $G$  any two straight lines  $GF$ ,  $GH$ , making the angle  $FGH$  equal to the angle  $BCD$ , and such that the angle  $FGH$  as seen from  $E$  is drawn in the same sense as the angle  $BCD$  as seen from  $A$ , and take  $GF$  equal to  $CB$ , and  $GH$  to  $CD$ .

Draw  $EF$ ,  $FH$ ,  $HE$ .



Because in the triangles  $FGH$ ,  $BCD$ ,

$FG$ ,  $GH$  are equal to  $BC$ ,  $CD$ ,

and the angle  $FGH$  to the angle  $BCD$ ,

the triangle  $FGH$  is equal to the triangle  $BCD$ , (I. Prop. 4.)

therefore it is possible to shift the figure  $EFGH$ ,

so that the triangle  $FGH$  will coincide with the triangle  $BCD$ .

If this be done,

because  $GE$  is at right angles to the plane  $FGH$ , and  $AC$  to the plane  $BCD$  and the angles  $FGH$ ,  $BCD$ , as seen from  $E$  and  $A$  respectively are drawn in the same sense,

$GE$  will coincide in direction with  $CA$ ,

and because  $GE$  is equal to  $CA$ ,  $E$  will coincide with  $A$ ,

and therefore the figure  $EFGH$  will coincide with the figure  $ABCD$ .

Therefore the trihedral angle  $E(FGH)$  coincides with the trihedral angle  $A(BCD)$  and is equal to it.

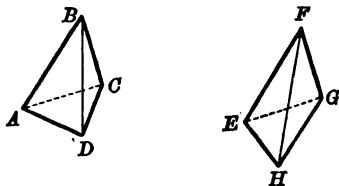
ADDITIONAL PROPOSITION 5.

If two plane angles of one trihedral angle be equal to two plane angles of another and the inclination of the faces containing the angles of the one be greater than the inclination of the faces containing the angles of the other, the third plane angle of the one is greater than the third plane angle of the other.

Let  $A(BCD)$ ,  $E(FGH)$  be two trihedral angles, in which the plane angles  $BAC$ ,  $CAD$  are equal to the plane angles  $FEG$ ,  $GEH$  respectively and the inclination of the planes  $BAC$ ,  $CAD$  is greater than the inclination of the planes  $FEG$ ,  $GEH$ .

In the corresponding edges  $AC$ ,  $EG$  take  $AC$  equal to  $EG$ ; and draw  $CB$ ,  $CD$  at right angles to  $CA$  to meet  $AB$ ,  $AD$  in  $B$  and  $D$ , and draw  $GF$ ,  $GH$  at right angles to  $GE$  to meet  $EF$ ,  $EH$  in  $F$  and  $H$ .

Draw  $BD$ ,  $FH$ .



Because the inclination of the planes  $BAC$ ,  $CAD$  is greater than the inclination of the planes  $FEG$ ,  $GEH$ ,

and  $CB$ ,  $CD$  are at right angles to  $CA$ ,  
and  $GF$ ,  $GH$  are at right angles to  $GE$ ,  
the angle  $BCD$  is greater than the angle  $FGH$ .

Because in the triangles  $BAC$ ,  $FEG$ ,

the angles  $BAC$ ,  $ACB$  are equal to the angles  $FEG$ ,  $EGF$ ,  
and  $AC$  is equal to  $EG$ ,

therefore  $AB$  is equal to  $EF$ , and  $BC$  to  $FG$ . (I. Prop. 26, Part 1.)

Similarly it can be proved that

$AD$  is equal to  $EH$  and  $DC$  to  $HG$ .

Because in the triangles  $BCD$ ,  $FGH$ ,

$BC$ ,  $CD$  are equal to  $FG$ ,  $GH$ ,  
and the angle  $BCD$  is greater than the angle  $FGH$ ,  
therefore  $BD$  is greater than  $FH$ . (I. Prop. 24.)

Because in the triangles  $BAD$ ,  $FEH$ ,

$BA$ ,  $AD$  are equal to  $FE$ ,  $EH$ ,  
and  $BD$  is greater than  $FH$ ,

therefore the angle  $BAD$  is greater than the angle  $FEH$ . (I. Prop. 25.)

## ADDITIONAL PROPOSITION 6.

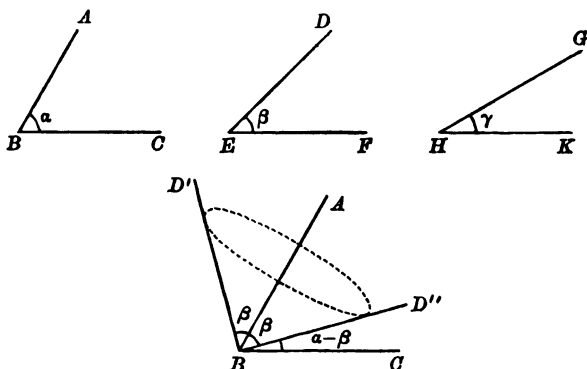
[Euclid, Elem. xi. Prop. 22.]

If the sum of three angles be less than four right angles, and if each angle be less than the sum of the other two, a trihedral angle can be constructed having its plane angles equal to the given angles.

Let  $ABC$ ,  $DEF$ ,  $GHK$  be three plane angles such that the sum of the angles is less than four right angles and each angle is less than the sum of the other two.

Let the angle  $ABC$  be not less than either of the angles  $DEF$ ,  $GHK$ , and the angle  $DEF$  be not less than the angle  $GHK$ .

Fig. 1.



Let the two planes  $ABC$ ,  $DEF$  be so placed that  $E$  coincides with  $B$  and  $EF$  in direction with  $BA$ . If now one of the planes be turned round  $BA$ , so that the inclination of the two planes changes continuously from zero to two right angles, the plane angle between  $BC$  and  $ED$  will change continuously from a minimum to a maximum (Add. Prop. 5).

(1) In the case, when the sum of the angles  $ABC$ ,  $DEF$  is not greater than two right angles, the maximum value is the sum of the angles  $ABC$ ,  $DEF$  (Fig. 1);

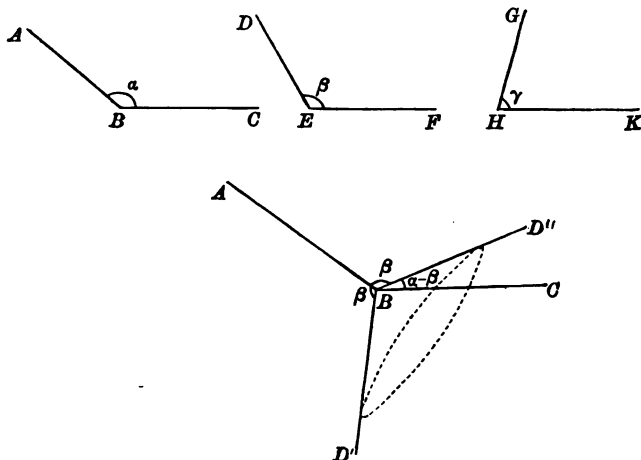
and (2) in the case, when the sum of the angles  $ABC$ ,  $DEF$  is greater than two right angles, the maximum value is the sum of their supplements (Fig. 2).

In both cases the minimum value is the difference of the angles  $ABC$ ,  $DEF$ .



In the first case the sum of the angles  $ABC$ ,  $DEF$  is greater than the angle  $GHK$  (Hypothesis).

Fig. 2.



In the second case because the sum of the angles  $ABC$ ,  $DEF$ ,  $GHK$  is less than four right angles,

the angle  $GHK$  is less than the sum of the supplements of the angles  $ABC$ ,  $DEF$ .

In both cases because  $ABC$  is less than the sum of the angles  $DEF$ ,  $GHK$ ,

the difference between the angles  $ABC$ ,  $DEF$  must be less than the angle  $GHK$ .

Therefore in both cases there must be a position of the planes in which the angle between  $BC$  and  $EF$  is equal to the angle  $GHK$ .

In every case therefore in which the given conditions hold it is possible to find a position of the planes in which the three lines  $BA$ ,  $BC$ ,  $EF$  are the edges of a trihedral angle having its plane angles equal to the given plane angles.

## ADDITIONAL PROPOSITION 7.

[Euclid, Elem. xi. Prop. 23.]

To construct a trihedral angle having its plane angles equal to three given plane angles.

Let  $ABC$ ,  $DEF$ ,  $GHK$  be three given plane angles.

Take  $BA$ ,  $BC$ ,  $ED$ ,  $EF$ ,  $HG$ ,  $HK$  all equal, and draw  $AC$ ,  $DF$ ,  $GK$ .

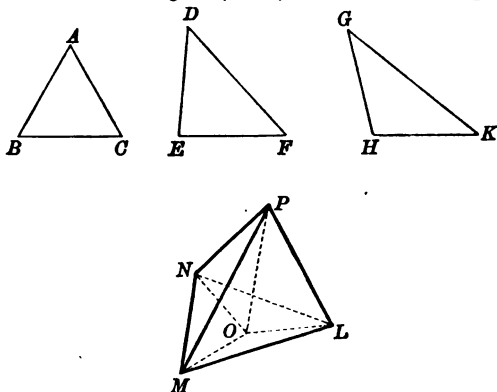
Construct a triangle  $LMN$  having its sides  $MN$ ,  $NL$ ,  $LM$  equal to  $AC$ ,  $DF$ ,  $GK$  respectively. (I. Prop. 23.)

Find  $O$  the centre of the circle  $LMN$ , and draw  $OL$ ,  $OM$ ,  $ON$ .

Draw through  $O$  the straight line  $OP$  at right angles to the plane  $LMN$ . (Prop. 12.)

With centre  $L$  and radius equal to  $BA$  draw a circle in the plane  $LOP$ ,

let it cut  $OP$  in  $P$ , and draw  $PL$ ,  $PM$ ,  $PN$  :  
then the trihedral angle  $P(LMN)$  is constructed as required.



Because  $OP$  is at right angles to the plane  $LMN$ , each of the angles  $LOP$ ,  $MOP$ ,  $NOP$  is a right angle.

Because in the triangles  $LOP$ ,  $MOP$ ,

$LO$ ,  $OP$  are equal to  $MO$ ,  $OP$ ,

and the angle  $LOP$  to the angle  $MOP$ ,

therefore  $LP$  is equal to  $MP$ ,

and therefore  $MP$  is equal to  $GH$  or  $KH$ . (I. Prop. 4.)

Because in the triangles  $LPM$ ,  $GKH$ ,

$LP$ ,  $PM$  are equal to  $GH$ ,  $HK$ ,

and  $LM$  to  $GK$ ,

therefore the angle  $LPM$  is equal to the angle  $GKH$ . (I. Prop. 8.)

Similarly it can be proved that the angle  $MPN$  is equal to the angle  $ABC$  and the angle  $NPL$  to the angle  $DEF$ .

It may be observed that it is not possible to construct a trihedral angle which shall have its plane angles equal to *any* three given plane angles. It has been proved in Prop. 20, that the sum of any two plane angles of a trihedral angle is greater than the third angle, and in Prop. 21, that the sum of the plane angles of a trihedral angle is less than four right angles. It follows therefore that it is impossible to construct a trihedral angle having its plane angles equal to three given angles, except when *the sum of any two of the given angles is greater than the third, and the sum of all the given angles is less than four right angles*. It has been proved in Add. Prop. 6 that, when these conditions exist, the construction of the trihedral angle is possible. It follows therefore that, if these conditions exist, the construction of the triangle  $LMN$  is possible and the circle used in the construction in Add. Prop. 7 must cut the line  $OP$ .

If in the construction of Add. Prop. 7 the perpendicular through  $O$  to the plane  $LMN$  had been drawn on the other side of the plane, a second point  $P'$  would be found such that the trihedral angle  $P'(LMN)$  had its plane angles equal to the given angles,  $ABC, DEF, GHK$  which we will call  $\alpha, \beta, \gamma$ . Here, though the plane angles of the trihedral angles  $P(LMN), P'(LMN)$  are equal in pairs, the trihedral angles themselves cannot be made to coincide and are therefore *not equal*. It will be proved afterwards (Add. Prop. 17) that they are *equal in opening*.

*Two trihedral angles, which have their plane angles equal in pairs but are not themselves equal, are called reverse trihedral angles.*

If from any point  $P$  a perpendicular  $PO$  be drawn on the plane of a triangle  $LMN$  and produced to  $P'$ , so that  $OP'$  is equal to  $OP$ , the pairs of trihedral angles  $P(LMN), P'(LMN); L(PMN), L(P'MN); M(PNL), M(P'NL); N(PLM), N(P'LM)$  are reverse.

In this case the two tetrahedrons  $P(LMN), P'(LMN)$  have their faces equal in pairs, but they cannot be made to coincide and are therefore *not equal*. It will be proved afterwards (Add. Prop. 19) that they are *equal in volume*.

*Two tetrahedrons, which have their faces equal in pairs but are not themselves equal, are called reverse tetrahedrons.*

The nature of the difference of two reverse trihedral angles may be realised by the following consideration.

If a spectator were to be placed within the trihedral angle  $P(LMN)$  with his head towards  $P$  and with his back towards the plane  $PMN$ , he would have the plane containing  $\alpha$  behind him, the plane containing  $\beta$  on his left, and the plane containing  $\gamma$  on his right; if on the other hand the spectator were to be placed in the trihedral angle  $P'(LMN)$  with his head towards  $P'$  and his back to the plane  $P'MN$ , he would have the plane containing  $\alpha$  behind him, the plane containing  $\beta$  on his right, and the plane containing  $\gamma$  on his left.

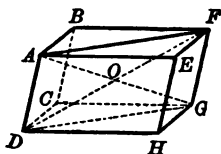
## ADDITIONAL PROPOSITION 8.

[Euclid, Elem. xi. Prop. 24.]

*In a parallelepiped, all the faces are parallelograms, and the straight lines joining opposite corners\* meet in a point and bisect each other.*

Let  $ABCDEFGH$  be a parallelepiped.

Draw the straight lines  $AG$ ,  $GD$ ,  $DF$ ,  $FA$  joining the extremities of two opposite edges  $AD$ ,  $FG$ .



Because  $BH$  is a parallelepiped,  
the planes  $ABFE$ ,  $DCGH$  are parallel;  
and  $ABCD$  is a plane cutting them,  
therefore  $AB$  is parallel to  $DC$ . (Prop. 16.)

Similarly it can be proved that

$AD$  is parallel to  $BC$ .

Therefore  $ABCD$  is a parallelogram.

Similarly it can be proved that every face is a parallelogram.

Because  $ABCD$  is a parallelogram,

$AD$  is equal and parallel to  $BC$ ; (I. Prop. 33.)

and because  $FGCB$  is a parallelogram,

$FG$  is equal and parallel to  $BC$ ;

therefore  $AD$  is equal and parallel to  $FG$ ; (Prop. 9.)

therefore  $AF$  is equal and parallel to  $DG$ , (I. Prop. 33.)

that is,  $AFGD$  is a parallelogram:

therefore  $AG$ ,  $FD$  bisect each other at a point  $O$ . (See Ex. 1, p. 91.)

Similarly it can be proved that every pair of diagonals bisect each other; all the diagonals therefore meet at a point and bisect one another.

\* A straight line joining two opposite corners of a parallelepiped is called a **diagonal** of the parallelepiped.

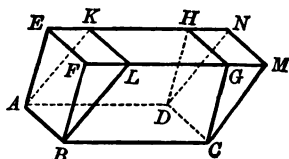
ADDITIONAL PROPOSITION 9.

[Euclid, Elem. xi. Props. 29, 30.]

*Two parallelepipeds, which have one face common and the faces opposite to the common face in a plane, are equal in volume.*

Let the parallelepipeds  $ABCDEFGH$ ,  $ABCDKLMN$  have the face  $ABCD$  common and the opposite faces  $EFGH$ ,  $KLMN$  in a plane.

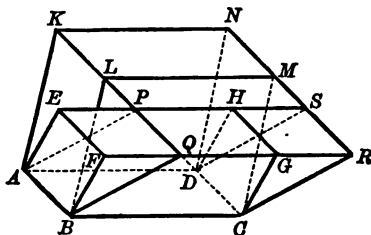
First, let the faces  $AH$ ,  $AN$  be in a plane and therefore the faces  $BG$ ,  $BM$  in a plane.



Because  $AG$ ,  $AM$  are parallelepipeds,  
 $EH$ ,  $KN$  are each equal to  $AD$ ; (Add. Prop. 8.)  
 therefore  $EK$  is equal to  $HN$ ,  
 also  $EA$  is equal and parallel to  $HD$   
 and  $EF$  is equal and parallel to  $HG$ .

It may easily be proved that the figures  $EAKFBL$ ,  $HDNGCM$  are equal in all respects;  
 therefore the volume of the parallelepiped  $AG$  is equal to that of  $AM$ .

Next let no two of the faces of the parallelepipeds be in a plane except  $ABCD$  and the faces parallel to it.



Let the lines  $FG$ ,  $EH$  produced cut  $LK$  in  $Q$ ,  $P$ , and  $MN$  in  $R$ ,  $S$ .  
 Draw  $AP$ ,  $BQ$ ,  $CR$ ,  $DS$ .

Then by the first case  
 the parallelepipeds  $ABCDEFGH$ ,  $ABCDKLMN$  are each equal in volume to the parallelepiped  $ABCDPQRS$ , and are therefore equal in volume to one another.

**COROLLARY.** *Two parallelepipeds, which have two faces equal in all respects and which are of the same altitude, are equal in volume.*

## ADDITIONAL PROPOSITION 10.

[Euclid, Elem. xi. Prop. 31.]

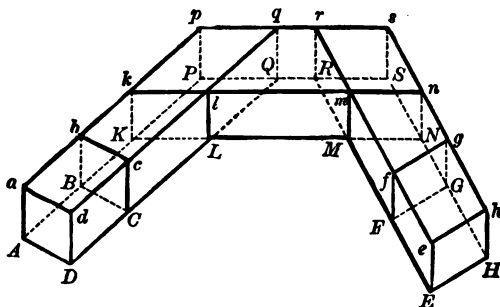
Two parallelepipeds, which have two faces equal in area and in a plane, and the faces opposite to these faces also in a plane, are equal in volume.

Let  $ABCDabcd$ ,  $EFGHefgh$  be two parallelepipeds whose faces  $ABCD$ ,  $EFGH$  are equal in area and in a plane, and whose faces  $abcd$ ,  $efgh$  are in a plane.

Let  $AB$ ,  $DC$  produced meet  $EF$ ,  $HG$  produced in  $I$ ,  $J$  respectively.

Draw  $KLMN$  parallel to  $IJ$  to cut  $AB$ ,  $DC$ ,  $EF$ ,  $HG$  in  $K$ ,  $L$ ,  $M$ ,  $N$  respectively.

On  $ABK$  take  $KP$  equal to  $AB$  and draw  $PQRS$  parallel to  $KLMN$  to cut  $DC$ ,  $EF$ ,  $HG$  in  $Q$ ,  $R$ ,  $S$ . Complete the figure, so that the edges of the parallelepipeds\*  $Aa$ ,  $Kk$ ,  $Mm$ ,  $Ee$  are parallel.



Because  $AB$  is equal to  $KP$ , and  $Aa$ ,  $Kk$  are parallel, the parallelograms  $AabB$ ,  $KkpP$  are equal in all respects; therefore the parallelepiped  $ABCDabcd$  is equal in volume to the parallelepiped  $KPQLkpql$ . (Add. Prop. 9, Coroll.)

Similarly it can be proved that the parallelepiped  $EFGHefgh$  is equal in volume to the parallelepiped  $MRSNmrsn$ ,

and because  $KL$  is equal to  $MN$  each being equal to  $IJ$  and  $Kk$ ,  $Mm$  are parallel,

the parallelograms  $KkIL$ ,  $MmnN$  are equal in all respects;

therefore the parallelepiped  $KPQLkpql$  is equal in volume to the parallelepiped  $MRSNmrsn$ ,

therefore the parallelepipeds  $ABCDabcd$ ,  $EFGHefgh$  are equal in volume.

\* This assumes that the edges  $Aa$ ,  $Ee$  are parallel. This does not destroy the generality of the proof as is shewn by Add. Prop. 9.

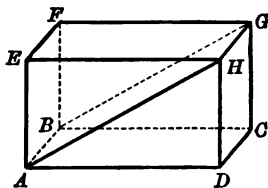
ADDITIONAL PROPOSITION 11.

[Euclid, Elem. xi. Prop. 28.]

*A rectangular parallelepiped is bisected by a plane through two opposite edges.*

Let  $ABCDEFGH$  be a rectangular parallelepiped.

Draw the plane  $ABGH$  cutting the faces  $AEHD$ ,  $BFGC$  in  $AH$ ,  $BG$ .



Because  $AE$  is equal to  $HD$ ,  $EH$  to  $DA$  and  $HA$  to  $AH$ ,

the triangles  $AEH$ ,  $HDA$  are equal in all respects. (I. Prop. 8.)

Therefore the figure  $AEHBFH$  can be shifted so that the triangle  $AEH$  coincides exactly with the triangle  $HDA$ .

If this be done, then because the lines  $AB$ ,  $EF$ ,  $HG$ ,  $DC$  are all at right angles to the plane  $AEHD$ , (Hypothesis)

the lines  $AB$ ,  $EF$ ,  $HG$  after shifting will coincide in direction with  $HG$ ,  $DC$ ,  $AB$  respectively,

and because  $AB$ ,  $EF$ ,  $HG$ ,  $DC$  are all equal, the points  $B$ ,  $F$ ,  $G$  after shifting will coincide with the points  $G$ ,  $C$ ,  $B$  respectively,

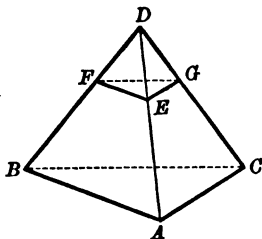
and therefore the figure  $AEHBFH$  will coincide with the figure  $HDAGCB$  and be equal to it.

The orthohedron is therefore divided into two exactly equal parts by the plane  $ABGH$ .

## ADDITIONAL PROPOSITION 12.

*Sections of a tetrahedron by planes parallel to the base are similar to the base.*

Let  $ABCD$  be a tetrahedron, and let  $EFG$  be the section by a plane parallel to the plane  $ABC$ .



Because the parallel planes  $ABC$ ,  $EFG$  are cut by the plane  $EFBA$ ,

$AB$  is parallel to  $EF$ . (Prop. 16.)

Similarly it can be proved that

$BC$  is parallel to  $FG$ .

Therefore the angle  $ABC$  is equal to the angle  $EFG$ . (Prop. 10.)

Similarly it can be proved that

the angle  $BCA$  is equal to the angle  $FGE$ ;

therefore the triangles  $ABC$ ,  $EFG$  are equiangular and similar.

(VI. Prop. 4.)

**COROLLARY.** *Sections of a pyramid by planes parallel to the base are similar to the base.*

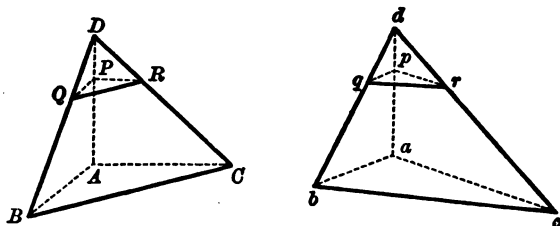


ADDITIONAL PROPOSITION 13.

*The sections of two tetrahedrons on equal bases made by planes parallel to the bases and dividing corresponding edges proportionally are equal in area.*

Let  $ABCD$ ,  $abcd$  be two tetrahedrons on equal bases  $ABC$ ,  $abc$ , and let  $PQR$ ,  $pqr$  be two sections made by planes parallel to the bases dividing the edges  $AD$ ,  $ad$  proportionally; that is, so that

$$AP : PD \text{ as } ap : pd.$$



Because the triangles  $ABC$ ,  $PQR$  are similar, the ratio of the triangle  $ABC$  to the triangle  $PQR$

is duplicate of the ratio  $AB$  to  $PQ$ ; (VI. Prop. 19.)

and the ratio of the triangle  $abc$  to the triangle  $pqr$

is duplicate of the ratio  $ab$  to  $pq$ ;

and  $AB$  is to  $PQ$  as  $AD$  to  $PD$ ;

$ab$  is to  $pq$  as  $ad$  to  $pd$ ;

and  $AD$  is to  $PD$  as  $ad$  to  $pd$ ;

therefore  $AB$  is to  $PQ$  as  $ab$  to  $pq$ ;

therefore the triangle  $ABC$  is to the triangle  $PQR$  as the triangle  $abc$  to the triangle  $pqr$ ;

and the triangle  $ABC$  is equal to the triangle  $abc$ ;

therefore the triangle  $PQR$  is equal to the triangle  $pqr$ .

## ADDITIONAL PROPOSITION 14.

[Euclid, Elem. xii. Prop. 5.]

*Tetrahedrons on equal bases and of equal altitudes are equal in volume.*

Let  $ABCD$ ,  $abcd$  be two tetrahedrons on equal bases  $ABC$ ,  $abc$  and of equal altitudes.

First, let the altitudes of the tetrahedrons be the edges  $AD$ ,  $ad$ .

Divide  $DA$ ,  $da$  each into the same number of equal parts

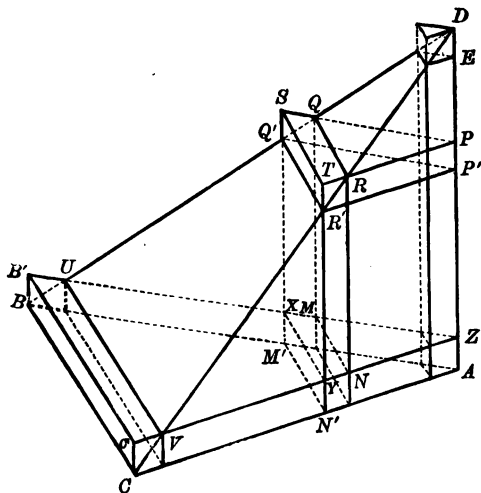
$DE, \dots PP', \dots ZA$ ;  $de, \dots pp', \dots za$ .

Through  $P$ ,  $P'$ , ...,  $Z$  draw planes parallel to the base  $ABC$ .

Let the section of the tetrahedron by the planes through  $P$ ,  $P'$ ,  $Z$  be the triangles  $PQR$ ,  $P'Q'R'$ ,  $ZUV$ .

Let a plane through  $QR$  parallel to  $AD$  cut the plane  $ZUV$  in  $MN$ , and a plane through  $Q'R'$  parallel to  $AD$  cut the planes  $PQR$ ,  $ZUV$ ,  $ABC$  in  $ST$ ,  $XY$ ,  $M'N'$ .

Make the same construction for the tetrahedron  $abcd$ .



The volume of the slice  $PQR P'Q'R'$  of the tetrahedron is less than the volume of the prism  $PST P'Q'R'$

by  $QS Q'R'TR'$ ,

which is equal to  $MM'N'N'$ .

Therefore the volume of the tetrahedron  $ABCD$  is less than the sum of the volumes of the prisms  $PST P'Q'R'$

by part of the volume of the bottom prism  $ZB'C'ABC$ .

Similarly it can be proved that the volume of the tetrahedron  $abcd$  is less than the sum of the volumes of the prisms  $pst p'q'r'$

by part of the volume of the bottom prism  $zb'c'abc$ .

Because the volume of the prism  $PSTP'Q'R'$

is equal to the volume of the prism  $pstp'q'r'$ ;

(Add. Props. 10 and 11.)

therefore the sum of the prisms  $PSTP'Q'R'$

is equal to the sum of the prisms  $pstp'q'r'$ ;

and the bottom prisms  $ZB'C'ABC$ ,  $zb'c'abc$  are equal;

(Add. Props. 10 and 11.)

therefore the difference of the tetrahedrons  $ABCD$ ,  $abcd$  is less than either of the two bottom prisms;

and as these can be made as small as we please by taking the number of slices large enough, it follows that the volumes of the tetrahedrons  $ABCD$ ,  $abcd$  are equal.

Next, let the foot of the perpendicular from  $D$  on the face  $ABC$  fall within the face at  $H$ .

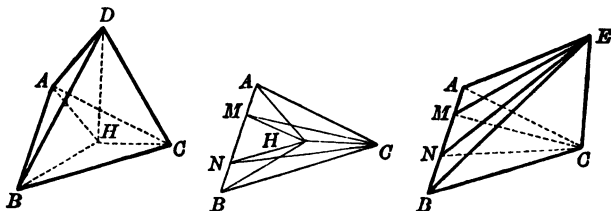
Draw  $HA$ ,  $HB$ ,  $HC$ ,

and draw  $HM$ ,  $HN$  parallel to  $CA$ ,  $CB$  to meet  $AB$  in  $M$ ,  $N$ ;

and draw  $CM$ ,  $CN$ .

Draw  $CE$  at right angles to the plane  $ABC$  and equal to  $DH$ ,

and draw  $EA$ ,  $EM$ ,  $EN$ ,  $EB$ .



Because the triangle  $HAC$  is equal to the triangle  $CAM$ ,

and the triangle  $HBC$  to the triangle  $CBN$ ,

therefore the triangle  $HAB$  is equal to the triangle  $CMN$ .

Therefore, by the first case,

the tetrahedrons  $D(CHA)$ ,  $D(AHB)$ ,  $D(BHC)$  are equal in volume to the tetrahedrons  $E(ACM)$ ,  $E(MCN)$ ,  $E(NCB)$  respectively;

and therefore the tetrahedron  $D(ABC)$  to the tetrahedron  $E(ABC)$ .

In the same way, when the foot of the perpendicular  $DH$  lies outside the triangle  $ABC$ , if the same construction be made, it can be proved that the tetrahedrons are equal in pairs as in the last case: but that the tetrahedron  $D(ABC)$  is equal to the difference between two of the tetrahedrons  $D(CHA)$ ,  $D(AHB)$ ,  $D(BHC)$  and the third; and that the tetrahedron  $E(ABC)$  is equal to the difference between the corresponding two of the tetrahedrons  $E(ACM)$ ,  $E(MCN)$ ,  $E(NCB)$  and the third.

**COROLLARY.** *Pyramids on equal bases and of equal altitudes have equal volumes.*

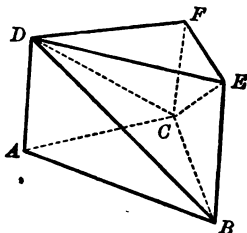
## ADDITIONAL PROPOSITION 15.

[Euclid, Elem. xii. Prop. 7.]

*A triangular prism whose end faces are parallel can be divided into three tetrahedrons of equal volumes.*

Let  $ABCDEF$  be a triangular prism of which the faces  $ABC$ ,  $DEF$  are parallel.

Draw  $BD$ ,  $DC$ ,  $CE$ .



Because  $ABED$  is a parallelogram,

the triangle  $ABD$  is equal to the triangle  $BED$ ;

therefore the tetrahedrons  $CABD$ ,  $CBED$  are equal in volume.

Because  $FCBE$  is a parallelogram, (Add. Prop. 14.)

the triangle  $ECF$  is equal to the triangle  $ECB$ ;

therefore the tetrahedrons  $DECF$ ,  $DECB$  are equal in volume;

therefore the tetrahedrons  $ABCD$ ,  $BCDE$ ,  $CDEF$  are equal in volume, and they together make up the prism  $ABCDEF$ .

**COROLLARY 1.** *The volume of a tetrahedron is equal to one third the volume of a prism on the same base and of the same altitude.*

**COROLLARY 2.** *Prisms on equal bases and of equal altitudes have equal volumes.*

**COROLLARY 3.** *The plane through two opposite edges of a parallelepiped divides the parallelepiped into two prisms of equal volume.*

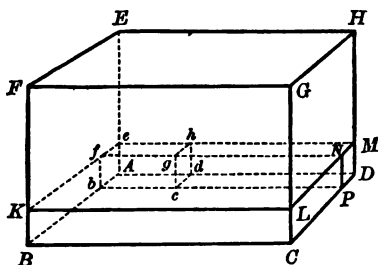
**NOTE.** It may be proved by using the method of proof adopted in Prop. 1 of Book VI. that the volumes of pyramids or parallelepipeds on equal bases are in the same ratio as their altitudes, and that the volumes of pyramids or parallelepipeds of equal altitudes are in the same ratio as the area of their bases.

ADDITIONAL PROPOSITION 16.

If two parallelepipeds have a solid angle of the one equal to a solid angle of the other, the ratio of the volumes of the parallelepipeds is equal to the ratio compounded of the ratios of the edges about the equal angles.

Let  $ABCDEFGH$ ,  $abcdefgh$  be two parallelepipeds having the trihedral angles  $A$  ( $BDE$ ),  $a$  ( $bde$ ) equal in all respects.

Let the figures be so placed that the point  $a$  may coincide with  $A$ , and the edges  $ab$ ,  $ad$ ,  $ae$  in direction with  $AB$ ,  $AD$ ,  $AE$  respectively. Let the plane  $efgh$  cut the edges  $BF$ ,  $CG$ ,  $DH$  in  $K$ ,  $L$ ,  $M$  respectively, and the edge  $fg$  cut  $LM$  in  $N$ , and  $bc$  cut  $CD$  in  $P$ .



Because the parallelepiped  $Abcdefgh$  is to the parallelepiped  $AbPDefNM$  as  $Ad$  to  $AD$ , (See Note, p. 570.)

and the parallelepiped  $AbPDefNM$  is to the parallelepiped  $ABCD eKLM$  as  $Ab$  to  $AB$ ,

and the parallelepiped  $ABCD eKLM$  is to the parallelepiped

$ABCDEFGH$  as  $Ae$  to  $AE$ ;

therefore the parallelepiped  $Abcdefgh$  has to the parallelepiped  $ABCDEFGH$  the ratio compounded of the ratios  $Ab$  to  $AB$ ,  $Ad$  to  $AD$  and  $Ae$  to  $AE$ .

**COROLLARY 1.** Similar parallelepipeds are in the ratio triplicate of the ratio of two corresponding edges. [Euclid, Elem. xi. Prop. 33.]

**COROLLARY 2.** If two tetrahedrons have a solid angle of the one equal to a solid angle of the other, the ratio of the volumes of the tetrahedrons is equal to the ratio compounded of the ratios of the edges about the equal angles.

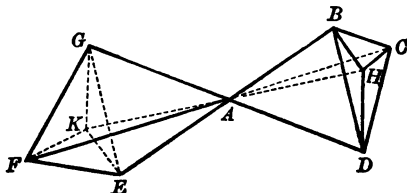
**COROLLARY 3.** Similar tetrahedrons are in the ratio triplicate of the ratio of two corresponding edges. [Euclid, Elem. xii. Prop. 8.]

## ADDITIONAL PROPOSITION 17.

*Two opposite vertical trihedral angles are equal in opening.*

Let  $A(BCD)$  be any trihedral angle, and let the straight lines  $BA$ ,  $CA$ ,  $DA$  be produced to  $E$ ,  $F$ ,  $G$  respectively.

Take  $AB$ ,  $AC$ ,  $AD$ ,  $AE$ ,  $AF$ ,  $AG$  all equal. Draw the planes  $BCD$ ,  $EFG$ ; they are parallel. (Prop. 15.) Draw  $HAK$  at right angles to the planes  $BCD$ ,  $EFG$  and draw  $HB$ ,  $HC$ ,  $HD$ ,  $KE$ ,  $KF$ ,  $KG$ .



Because the triangles  $BCD$ ,  $EFG$  are equal in all respects, and  $H$ ,  $K$  are the centres of the circles  $BCD$ ,  $EFG$  respectively, (See Proof of Add. Prop. 7.) therefore  $HB$ ,  $HC$ ,  $HD$ ,  $KE$ ,  $KF$ ,  $KG$  are equal.

Also  $AH$  is equal to  $AK$ .

The angle  $BHD$  is equal to the angle  $EKG$  and  $HA$ ,  $KA$  are at right angles to the planes of those angles;

therefore the trihedral angles  $H(ABD)$ ,  $K(AGE)$  are equal in all respects.

And  $HA$ ,  $HB$ ,  $HD$  are equal to  $KA$ ,  $KG$ ,  $KE$  respectively; therefore the tetrahedron  $ABHD$  is equal in all respects to the tetrahedron  $AGKE$ ,

and therefore the trihedral angle  $A(BHD)$  to the trihedral angle  $A(GKE)$ .

Similarly it can be proved that

the trihedral angles  $A(DHC)$ ,  $A(CHB)$  are equal in all respects to the trihedral angles  $A(GKF)$ ,  $A(EKF)$  respectively.

If  $H$  lie within the triangle  $BCD$ ,

then the trihedral angle  $A(BCD)$  is equal to the sum of the three trihedral angles  $A(BHD)$ ,  $A(DHC)$ ,  $A(CHB)$ .

But if  $H$  lie outside the triangle  $BCD$ ,

the trihedral angle  $A(BCD)$  is made up of the difference of two of these angles and the third,

and the trihedral angle  $A(EFG)$  is the corresponding sum or difference of the equal angles  $A(GKE)$ ,  $A(EKF)$ ,  $A(FKG)$ .

Hence the trihedral angles  $A(BCD)$ ,  $A(EFG)$  are equal in opening.

ADDITIONAL PROPOSITION 18.

To describe a sphere about a given tetrahedron.

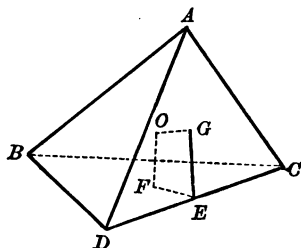
Let  $ABCD$  be a given tetrahedron.

Bisect any edge  $CD$  in  $E$ .

In the planes  $BCD$ ,  $ACD$  draw  $EF$ ,  $EG$  at right angles to  $CD$ .

Find  $F$ ,  $G$  the centres of the circles  $BCD$ ,  $ACD$ . (IV. Prop. 5.)

In the plane  $FEG$  draw  $FO$ ,  $GO$  at right angles to  $EF$ ,  $EG$  respectively.



Because  $F$  is the centre of the circle  $BCD$ , and  $FO$  is at right angles to the plane  $BCD$ , every point in  $FO$  is equidistant from  $B$ ,  $C$ ,  $D$ . (See Proof of Add. Prop. 7.)

Similarly it can be proved that every point in  $GO$  is equidistant from  $A$ ,  $C$ ,  $D$ .

Therefore the point  $O$  where they meet is equidistant from  $A$ ,  $B$ ,  $C$ ,  $D$  and a sphere having  $O$  for centre and passing through any one of the points  $A$ ,  $B$ ,  $C$ ,  $D$  must pass through the others.

**COROLLARY.** The radii of spheres described about equal or reverse tetrahedrons are equal.

If constructions similar to that in the proposition be made for two equal or two reverse tetrahedrons  $ABCD$ ,  $abcd$ , the angles at  $F$ ,  $E$ ,  $G$  are equal to the angles at  $f$ ,  $e$ ,  $g$  (Add. Prop. 1.) and  $FE$ ,  $EG$  are equal to  $fe$ ,  $eg$ .

Therefore the quadrilaterals  $OFE G$ ,  $ofeg$  are equal in all respects. (Ex. 4, page 27.)

Therefore in equal triangles  $OED$ ,  $oed$ ,  $OD$  is equal to  $od$ .

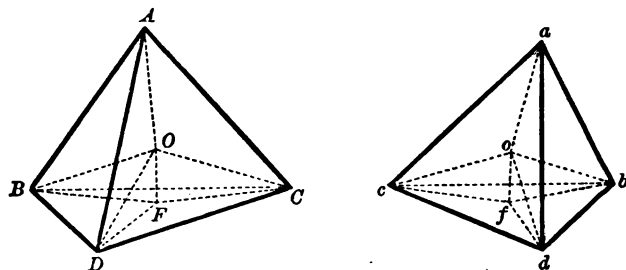
## ADDITIONAL PROPOSITION 19.

Two reverse tetrahedrons are either the sums or corresponding differences of twelve tetrahedrons which in pairs are equal in all respects.

Let  $ABCD$ ,  $abcd$  be two reverse tetrahedrons.

Find  $O$ ,  $o$  the centres of the spheres  $ABCD$ ,  $abcd$  (Add. Prop. 18.)  
and draw  $OF$ ,  $of$  perpendicular to the planes  $BCD$ ,  $bcd$ .

Draw  $OA$ ,  $OB$ ,  $OC$ ,  $OD$ ,  $FB$ ,  $FC$ ,  $FD$ ,  $oa$ ,  $ob$ ,  $oc$ ,  $od$ ,  $fb$ ,  $fc$ ,  $fd$ .



Because the tetrahedrons  $ABCD$ ,  $abcd$  are reverse, the radii of the spheres  $ABCD$ ,  $abcd$  are equal. (Add. Prop. 18, Coroll.)

In the tetrahedrons  $OCFD$ ,  $odfc$ ,  
the trihedral angles  $F( OCD)$ ,  $f( odc)$  are equal in all respects and  
 $FO$ ,  $FC$ ,  $FD$  equal to  $fo$ ,  $fd$ ,  $fc$  respectively;

therefore the tetrahedrons  $OCFD$ ,  $odfc$  are equal in all respects.

Similarly it can be proved that  
the tetrahedrons  $ODFB$ ,  $OBFC$  are equal in all respects to the tetrahedrons  $obfd$ ,  $ocfb$  respectively.

If the point  $F$  lie within the triangle  $BDC$  then the tetrahedron  $OBCD$  is the sum of the three tetrahedrons  $OCFD$ ,  $ODFB$ ,  $OBFC$ ,

but if the point  $F$  lie without the triangle  $BDC$  the tetrahedron  $OBCD$  is the difference between two of these tetrahedrons and the third;

and similarly the tetrahedron  $obcd$  is the corresponding sum or difference of the tetrahedrons  $odfc$ ,  $ocfb$ ,  $obfd$ .



Therefore the tetrahedrons  $OBCD$ ,  $obcd$  are equal in volume; they are each the sum or the difference of three tetrahedrons which in pairs are equal in all respects.

Similarly it can be proved that the same property belongs to the two tetrahedrons  $OCDA$ ,  $ocda$ ; likewise to the two tetrahedrons  $ODAB$ ,  $odab$ ; and likewise to the two tetrahedrons  $OABC$ ,  $oabc$ .

If the point  $O$  lie within the tetrahedron  $ABCD$ , then the tetrahedron  $ABCD$  is the sum of the four tetrahedrons  $OBCD$ ,  $OCDA$ ,  $ODAB$ ,  $OABC$ , but if the point  $O$  lie without the tetrahedron  $ABCD$ , then the tetrahedron  $ABCD$  is the difference between three of these tetrahedrons and the fourth, or between two of them and the other two;

and similarly the tetrahedron  $abcd$  is the corresponding sum or difference of the tetrahedrons  $obcd$ ,  $ocda$ ,  $odab$ ,  $oabc$ .

Therefore the whole tetrahedrons  $ABCD$ ,  $abcd$  are equal in volume; they are the corresponding sum or difference of twelve tetrahedrons which in pairs are equal in all respects. In particular, when all the faces are acute-angled triangles, the points such as  $F$ ,  $f$  all lie within the corresponding triangles, and the points  $O$ ,  $o$  lie within the tetrahedrons  $ABCD$ ,  $abcd$  respectively; and then the reverse tetrahedrons are divided each into twelve tetrahedrons which in pairs are equal in all respects.

## PERSPECTIVE.

1. Without attempting to discuss what light is or what the physical means are by which light is propagated, we may assume that light passes through space from one point to another in a straight line. The straight lines in which light travels are called rays of light. If we assume this rectilinear transmission as a fact, we may say that the light proceeding from a point produces the same impression on an eye as if it proceeded from any other point in the straight line joining the luminous point to the eye. In this statement we take no notice of the amount of light or of the quality of the light, but merely of the direction in which the luminous point appears visible to the eye.

2. If therefore on a fixed plane we marked the points in which straight lines joining a series of luminous points to the eye cut the plane, light proceeding from those points would produce the same impression on the eye as light proceeding from the luminous points themselves.

3. Such a plane is called a **picture plane**, and our object now is to discuss the geometrical relations between the positions of the luminous points themselves and their representative points on the picture plane.

4. When a series of luminous points lies in a straight line, the rays of light from each of the points to the eye will lie in the plane which can be drawn through the eye and the straight line in which the luminous points lie. The representative points therefore in the picture plane will lie in the straight line in which the picture plane is cut by the plane through the eye and the luminous points. (Prop. 1.) When a series of luminous points lies in a straight line whose direction passes through the eye, all the points of the series are represented by the same point in the picture plane, the point in which the picture plane is cut by the line in which the luminous points lie.

5. If two straight lines in space intersect, they will be represented in the picture plane by two straight lines which intersect in the point

representing their point of intersection. It must not be assumed, however, that when two straight lines in the picture plane intersect the straight lines in space which are represented by the lines in the picture plane necessarily intersect. It is always possible to draw from the eye a straight line to intersect two nonintersecting straight lines in space; this straight line will cut the picture plane in the point in which the representative straight lines intersect.

6. If two straight lines in space be parallel to one another and also parallel to the picture plane, they will be represented by two straight lines in the picture plane parallel to them. (Props. 9 and 16.) Equal straight lines lying in a plane parallel to the picture plane are represented in the picture by equal straight lines parallel to them. (Prop. 16 and VI. Prop. 4.) Two lines in space which are parallel to the picture plane but not parallel to each other will be represented in the picture plane by two lines that intersect. Straight lines which are represented in the picture by parallel straight lines are not necessarily parallel. It will be seen that straight lines lying in planes which intersect in a straight line through the eye parallel to the picture plane will be represented in the picture by parallel straight lines; and that if two straight lines in space be parallel to one another but not parallel to the picture plane, they will be represented by two straight lines which intersect in the point in which a straight line parallel to them through the eye cuts the picture plane. (Prop. 3.)

7. The straight line which is drawn from the eye at right angles to the picture plane is called the *line of sight*, and the point where this straight line cuts the picture plane is called the *centre of the picture*. As a rule the picture plane is supposed to be a vertical plane. Then a horizontal line drawn through the centre of the picture is called the *horizon line*.

8. In this case any series of parallel horizontal lines in space, which are not parallel to the picture plane, is represented in the picture plane by a series of lines converging to some point in the horizon line.

9. Any point in the picture plane to which the representative lines of any series of parallel lines converge is called the *vanishing point* of the parallel lines.

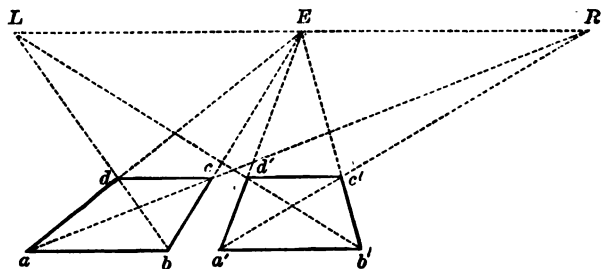
10. The vanishing point of any series of lines parallel to a horizontal plane is obtained thus; draw  $EV$  coinciding with the horizon line, draw  $EO$  at right angles to  $EV$ , make  $EO$  equal to the distance of the eye from the picture plane and draw  $OV$ , making the angle  $EOV$  equal to the angle between the line of sight and one of the parallel lines. The same construction will give the vanishing point for any series of parallel lines, if  $EV$  be the line of intersection of the picture plane with the plane passing through the line of sight and one of the parallel lines.

11. The vanishing points for all straight lines parallel to any plane through the line of sight lie on a straight line through the centre of the picture, and the vanishing points for any three directions mutually at right angles to one another form a triangle of which the centre of the picture is the orthocentre. (Exercise 5, p. 529.)

12. All lines parallel to the picture plane are represented by lines parallel to themselves, and as all vertical lines in space are parallel to the picture plane, they are represented by vertical lines in the picture plane.

13. The general problem in perspective drawing is to find on the picture plane the representative points of any series of points in space whose position relative to the eye is known.

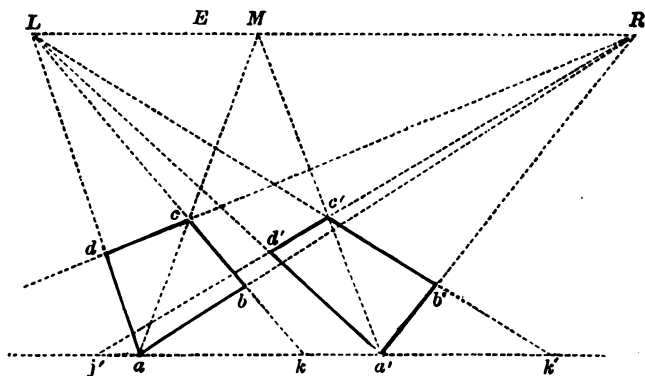
14. As an illustration of the use of these principles let us take first the case of a square  $ABCD$ , whose plane is horizontal and whose edge  $AB$  is parallel to the picture plane.



Let  $a, b, c, d$  be the points in the picture plane which represent the points  $A, B, C, D$ . In the picture  $ab, cd$  will be horizontal lines

(§ 6), the lines  $ad$ ,  $bc$  will pass through the centre of the picture  $E$  (§§ 8, 9). The diagonal  $AC$  makes equal angles of  $45^\circ$  with  $AB$  and  $AD$ ,  $ac$  therefore must cut the horizontal line at a definite point  $R$  (§ 8), where  $ER$  is equal to the distance from  $E$  to the eye (§ 10), if therefore  $ab$  represent the side  $AB$  we can complete the drawing thus; draw  $Ea$ ,  $Eb$ ,  $Ra$ ; let  $Ra$ ,  $Eb$  meet at  $c$ , draw  $cd$  parallel to  $ba$ , then  $abcd$  represents  $ABCD$ .

15. Next take the case of a horizontal square  $ABCD$  whose corner  $A$  is the nearest to the picture plane.



In this case  $ab$ ,  $dc$  will converge to some point  $R$  in the horizontal line on the right-hand side of  $E$  (§ 8);  $ad$ ,  $bc$  will converge to another point  $L$  in the horizontal line on the left-hand side of  $E$ ;  $R$  and  $L$  are such that a circle on  $RL$  as diameter would pass through the eye  $O$ , since  $OR$  and  $OL$  are parallel to  $AB$  and  $AD$  and therefore at right angles to each other. The diagonal  $AC$  bisects the angle between  $AB$ ,  $AD$ ,  $ac$  therefore must cut the horizontal line in a point  $M$  where  $OM$  is the bisector of the angle  $ROL$  (§ 8). Now let  $ab$  represent  $AB$ , draw  $La$ ,  $Lb$ ,  $Ma$ ; let  $Ma$ ,  $Lb$  cut in  $c$ , draw  $cd$  passing through  $R$ ; then  $abcd$  represents  $ABCD$ .

In both figures  $a'b'c'd'$  represents another position of  $ABCD$ .

16. The more definite problem of representing a figure of given size and in a given position on a picture plane in a given position is illustrated by the following example. A given cube  $ABCD FGHK$  whose edge is 4 inches, standing on a horizontal plane 1 foot below the eye, the nearest edge  $AF$  being 6 feet in front of the eye and 1 foot to the right of the line of sight, and the vertical faces being equally inclined to the line of sight, is to be represented on a vertical picture plane at a distance of 2 feet from the eye.

Through  $E$  the centre of the picture draw the horizon line  $LER$ , and take  $ER, EL$  each equal to  $OE$ , the distance from  $O$ , the eye, to the picture plane; in  $ER$  take  $M$  such that  $EM$  is to  $ER$  (or  $EO$ ) as the distance of the edge  $AF$  to the right of the line of sight is to the distance of  $AF$  in front of the eye, i.e. 1 : 6, and draw  $Mfa$  vertical; every point in  $AF$  has its representative point in  $af$ .

In  $Ma$  take  $a$  such that  $Ma$  is to  $ER$  (or  $EO$ ) as the vertical distance of  $A$  below the eye is to the distance of  $A$  in front of the eye, i.e. 1 : 6; then  $a$  represents  $A$ . Draw  $La, Ra$ .

In  $Ma$  take  $f$  such that  $Mf$  is to  $ER$  as  $\frac{1}{2}$  is to 6; then  $f$  represents  $F$ .

Draw a line  $yax$  through  $a$  parallel to the horizon line. Every straight line in a plane drawn through  $AF$  parallel to the picture plane will be represented in the picture plane by a line parallel to itself having the same ratio to  $af$  as the line itself has to  $AF$ . Let a line through  $A$  parallel to the horizon line cut a line through  $D$  at right angles to  $AD$  in  $Y$ . Now take  $ay$  so that  $ay : af = AY : AD = AC : AD$ , and take  $ax$  equal to  $ay$ ; then draw  $Ry, Lx$ , intersecting  $La, Ra$  in  $d$  and  $b$  and each other in  $c$ .

Draw  $Lf, Rf$ , and let them meet lines at right angles to the horizon line through  $d$  and  $b$  in  $k$  and  $g$ ; draw  $Lg$  and  $Rk$  meeting in  $h$ ; draw  $hc$  which will be vertical; then  $abcd fghk$  will represent  $ABCD FGHK$ .

In drawing to scale it is always most convenient to select, as has been done here, some one point in the picture as a point of reference (as  $a$ ), and to calculate all lengths by reference to lines drawn through this point.

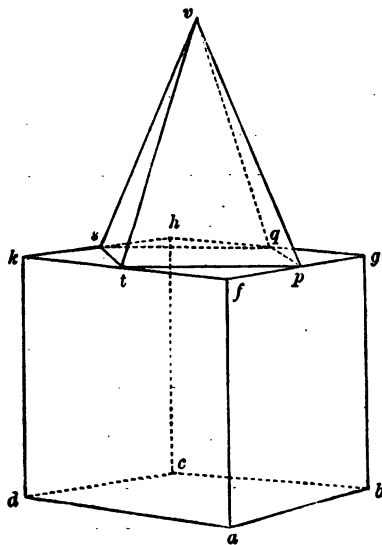
Let us assume now a pyramid  $VPQST$  whose height  $VN$  is four inches standing on the square base  $PQST$ , where  $P, Q, S, T$  are the middle points of  $FG, GH, HK, KF$ . The proof of the following con-

struction for the representation of the pyramid in the picture plane is left as an exercise for the student.

Draw  $fh$  which will cut the horizon line in  $E$ ; draw  $gk$  cutting  $fh$  in  $n$ ; draw  $Lsnp$  and  $Rqnt$ ; in  $af$  produced take  $fw$  equal to  $af$ ; draw  $fw$ , let it meet the vertical through  $n$  in  $v$ ; join  $pq$ ,  $qs$ ,  $st$ ,  $tp$ ,  $vp$ ,  $vg$ ,  $vs$ ,  $vt$ ; then  $vpqst$  will represent the pyramid  $VPQST$ . The complete perspective drawing of the cube surmounted by the pyramid is exactly represented below.

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*Horizon line*

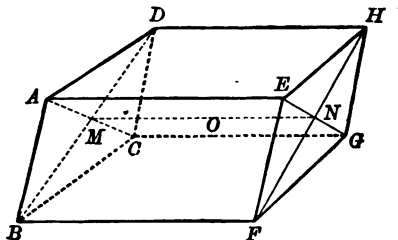


## GENERAL PROPERTIES OF A PARALLELEPIPED.

Let the diagonals  $AC$ ,  $BD$  of the face  $ABCD$  of a parallelepiped intersect in  $M$ , and the diagonals  $EG$ ,  $FH$  of the opposite face  $EFGH$  intersect in  $N$ ;

$MN$  is equal and parallel to the edges  $AE$ ,  $BF$ ,  $CG$ ,  $DH$ ;

Also  $MN$  passes through the centre  $O$  of the parallelepiped and is bisected there. (Add. Prop. 8.)



Let us consider the tetrahedron  $ACFH$ , a pair of whose opposite edges are  $AC$ ,  $FH$  diagonals of the opposite faces  $ABCD$ ,  $EFGH$  of the parallelepiped.

The straight line  $MN$ , which joins the middle points  $M$ ,  $N$  of  $AC$ ,  $FH$  a pair of its opposite edges, passes through  $O$  and is bisected there.

The other pairs of opposite edges of the tetrahedron  $AF$ ,  $CH$  and  $AH$ ,  $CF$  are also diagonals of opposite faces of the parallelepiped.

Hence as before the straight lines joining the middle points of the pairs of opposite edges of the tetrahedron,  $AF$ ,  $CH$  and  $AH$ ,  $CF$  are equal and parallel to  $AD$  and  $AB$  respectively, and pass through  $O$  and are bisected there.

When we take the tetrahedron, a pair of whose opposite edges are the diagonals  $BD$ ,  $EG$ , we find that the tetrahedron  $GEDB$  possesses the same properties as we have shewn to exist in the tetrahedron  $ACFH$ .

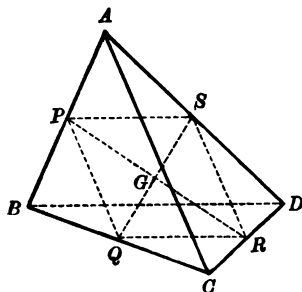
It may be observed that these two allied tetrahedrons  $ACFH$ ,  $GEDB$  have their corresponding points at opposite corners of the parallelepiped.



## GENERAL PROPERTIES OF A TETRAHEDRON.

1. Let  $ABCD$  be a tetrahedron,  
and  $P, Q, R, S$  the middle points of the edges  $AB, BC, CD, DA$ .

$PQ$  and  $SR$  are each parallel to  $AC$ ,  
and therefore are parallel to one another. (Prop. 9.)



Similarly  $PS, QR$  are parallel.

Hence  $PR, QS$  meet at a point  $G$  and bisect one another.

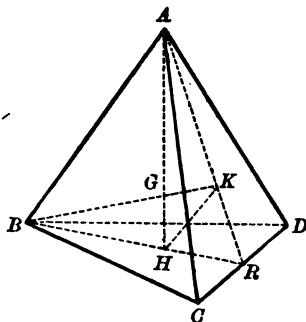
Similarly it can be proved that the line joining the middle points of  $AC$  and  $BD$  passes through  $G$  and is bisected there; hence the three straight lines joining the middle points of pairs of opposite edges of a tetrahedron meet at their middle points.

2. Draw  $RA$ ,  $RB$  and in them take points  $K$ ,  $H$ , such that  $AK$  is double of  $KR$  and  $BH$  is double of  $HR$ ,  $H$  and  $K$  are the centroids of the faces  $BCD$  and  $CDA$ .

Draw  $AH$ ,  $BK$ . Since they lie in the plane  $ARB$  they meet, let them meet at  $G$ .

Draw  $HK$ , then

$$AG : GH = BG : GK = AB : KH = AR : KR = 3 : 1.$$

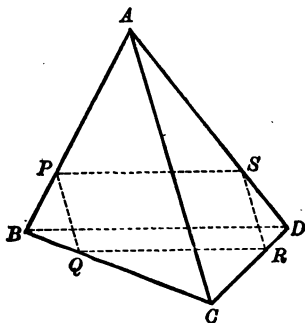


Similarly it can be shewn that the straight lines joining  $C$  and  $D$  to the centroids of the opposite faces meet  $AH$  or  $BK$  in  $G$  and are divided in the ratio of 3 to 1.

Hence the four straight lines which join the vertices of a tetrahedron to the centroids of the opposite faces meet at a point which divides them in the ratio 3 : 1.

It may be shewn in several ways, e.g. by the use of the Theorems of Ceva and Menelaus (see pages 422 and 418) or by the principle of the centroid of four equal weights (see page 425), that the points of intersection in the last two theorems are one and the same point.

3. Let a plane parallel to the opposite edges  $AC$ ,  $BD$  of a tetrahedron  $ABCD$  cut the edges  $AB$ ,  $BC$ ,  $CD$ ,  $DA$  in  $P$ ,  $Q$ ,  $R$ ,  $S$ , then  $PQ$



and  $SR$  are each parallel to  $AC$ , and therefore parallel to one another. (Prop. 9.) Similarly  $PS$  and  $QR$  are parallel; therefore  $PQRS$  is a parallelogram. Thus every section made by a plane parallel to two opposite edges is a parallelogram.

4. Now  $PQ$  is to  $AC$  as  $BP$  to  $BA$ ,  
and  $PS$  is to  $BD$  as  $AP$  to  $AB$ ;  
therefore if  $P$  be chosen so that

$AP$  is to  $PB$  as  $AC$  to  $BD$ ,

$PS$  is equal to  $PQ$ ,

and the section  $PQRS$  is a rhombus.

5. It may further be proved either by considering the centroid of four equal weights or otherwise that the locus of the centre of the section of a tetrahedron made by a plane parallel to two opposite edges is the straight line joining the middle points of those edges.

6. If through each of the edges of a tetrahedron a plane be drawn parallel to the opposite edge the six planes form a parallelepiped whose centre is the centroid of the tetrahedron and whose edges are equal and parallel to the straight lines joining the middle points of opposite edges of the tetrahedron.

The edges of the tetrahedron are the diagonals of opposite faces of the parallelepiped.

7. The volume of the parallelepiped is three times the volume of the tetrahedron.

### SPECIAL FORMS OF THE PARALLELEPIPED AND TETRAHEDRON.

1. If a parallelogram be a rhombus, its diagonals are at right angles and vice versa. If therefore a parallelepiped have one face a rhombus, the allied tetrahedrons have one pair of opposite edges at right angles, and vice versa. Again in a parallelepiped, if two pairs of opposite faces be rhombuses, the third pair are rhombuses; it follows then, if in a tetrahedron two pairs of opposite edges be at right angles, the third pair is also at right angles.

It follows that, if in a tetrahedron, one pair of opposite edges be at right angles, the straight lines joining the middle points of the other two pairs of opposite edges are equal, and vice versa.

2. If a parallelogram be a rectangle its diagonals are equal; and vice versa. Hence if a parallelepiped have one face a rectangle, each of the allied tetrahedrons has one pair of opposite edges equal; and vice versa. Also, if a parallelepiped have one face a rectangle, the straight lines joining the middle points of two of the pairs of opposite sides of each of the allied tetrahedrons are at right angles; and vice versa. Hence in a tetrahedron, if one pair of opposite edges be equal, the straight lines joining the middle points of the other two pairs of opposite edges are at right angles; and vice versa.

3. A parallelepiped can have one pair, two pairs, or three pairs of faces rectangular. Therefore a tetrahedron can have one pair, two pairs or three pairs of opposite edges equal.

If a tetrahedron have three pairs of opposite edges equal its faces are equal acute-angled triangles. (Ex. 1, page 547.)

4. In an orthohedron the square on a diagonal is equal to the sum of the squares on three concurrent edges.

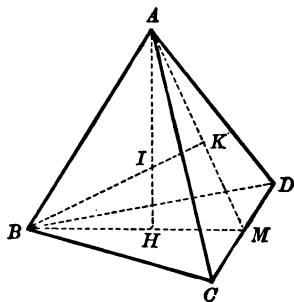
For in figure on page 565,

$$\begin{aligned} AG^2 &= AH^2 + HG^2 \\ &= AD^2 + DH^2 + HG^2 \\ &= AD^2 + AE^2 + AB^2. \end{aligned}$$

This result can also be stated thus:

The square on the distance between two points is equal to the sum of the squares on its projections on three straight lines at right angles to each other (see p. 533).

5. In a tetrahedron  $ABCD$ , let  $AH$ ,  $BK$  be drawn perpendicular to the faces  $BCD$ , and  $ACD$ . In general the lines  $AH$ ,  $BK$  will not meet, but for particular classes of tetrahedrons they may meet. Let them meet at  $I$  and let the plane  $AIB$  meet  $CD$  in  $M$ , then  $AKM$ ,  $BHM$  are straight lines. Now  $AH$  is at right angles to the plane  $BCD$ , and therefore at right angles to  $CD$ ; similarly  $BK$  is at right angles to  $CD$ , therefore  $CD$  is at right angles to the plane  $AIB$ , in which  $AH$ ,  $BK$  lie, and therefore to  $AB$ .



The converse is easily proved, viz. if  $AB$ ,  $CD$  be at right angles, the perpendiculars from  $A$  and  $B$  on the opposite faces intersect. Hence, if the perpendiculars from  $A$  and  $B$  on the opposite faces intersect, perpendiculars from  $C$  and  $D$  on the opposite faces also intersect.

Again if the perpendiculars  $AH$ ,  $BK$  intersect, since  $AKM$ ,  $BHM$  are each at right angles to  $CD$ , therefore

$$CA^2 - AD^2 = CM^2 - MD^2 = CB^2 - BD^2,$$

therefore

$$CA^2 + BD^2 = AD^2 + BC^2;$$

the converse of this is also true, viz. if  $CA^2 + BD^2 = AD^2 + BC^2$  then  $AB$ ,  $CD$  are at right angles.

Hence if three perpendiculars meet in a point, the fourth perpendicular passes through the same point, in which case,

$$AB^2 + CD^2 = AC^2 + BD^2 = AD^2 + BC^2.$$

6. A tetrahedron, in which the perpendiculars from the four vertices on the opposite faces meet in a point, is called an **orthocentric tetrahedron**; and the point, in which the perpendiculars meet, is called the **orthocentre** of the tetrahedron.

7. It can be proved that in a tetrahedron in which each pair of opposite edges is at right angles,

(1) the perpendicular from each vertex on the opposite face passes through the ortho-centre of that face;

(2) the middle points of the six edges lie on a sphere;

(3) the section of the sphere by each face is the Nine Point Circle of that face\*;

(4) the centre of the sphere is the middle point of the line joining the intersection of the perpendiculars from the vertices on the opposite faces, and the centre of the circumscribed sphere of the tetrahedron;

(5) the centre of the sphere coincides with the centroid of the tetrahedron.

\* This sphere is sometimes called the Twelve Point Sphere of the orthocentric tetrahedron.

## ROTATION.

The definitions of rotation about a point, which were adopted in Book III. (page 186) are equivalent to the following:—

*If a plane figure move in its plane unchanged in shape and size from any position to any other position, while one point in the figure remains fixed in the plane, the figure or any part of the figure is said to rotate about the fixed point.*

Similarly for motion in space:—

*If a figure move unchanged in shape and size from any position to any other position, while two points in the figure remain fixed in space, the figure or any part of the figure is said to rotate about the straight line joining the fixed points.*

## PROPERTIES OF THE SPHERE.

1. A sphere is defined as a closed surface which is such that all straight lines drawn to it from a fixed point are equal.

It follows that the section of a sphere made by a plane through the fixed point is a circle of which the fixed point is the centre, and as it has been found that a circle can have but one centre (III. Prop. 1), it follows that a sphere can have but one centre, and that centre is the common centre of all plane central sections.

**Definition.** *The section of a sphere by a plane drawn through the centre is called a great circle of the sphere.*

2. As two central planes must intersect in a diameter of the sphere, which is also a diameter of each circle, it follows that

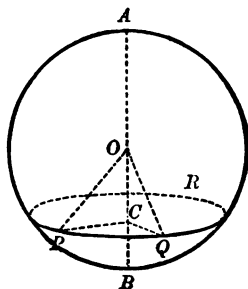
*Any two great circles bisect each other.*

If therefore a semicircle  $APB$  rotate round its diameter  $AB$  through four right angles, or the whole circle  $APB$  rotate through two right angles, it will generate a sphere of which  $AB$  is the diameter. Many of the common properties of a sphere are easily obtained by considering the sphere as generated by the rotation of a circle round a diameter, and many of the properties of two spheres can be obtained from properties of two circles by considering the spheres as generated by the rotation of two circles about the straight line joining their centres.



3. *Every plane section of a sphere is a circle.*

Let us take a plane section  $PQR$ , not passing through  $O$  the centre of the sphere. Draw the diameter  $AB$  to meet the plane  $PQR$  at right angles in  $C$ , and draw  $CP$ ,  $CQ$ ,  $OP$ ,  $OQ$ .



In the two right-angled triangles  $OCP$ ,  $OQ$ ,  
 $OP$ ,  $OQ$  are equal to  $OQ$ ,  $OQ$ ;  
 therefore  $CP$  is equal to  $CQ$ .

Similarly every other line from  $C$  to the section  $PQR$  is equal to  $CP$ .

Therefore the section  $PQR$  is a circle of which  $C$  is the centre.

**Definition.** *The section of a sphere made by a plane not passing through the centre is called a small circle of the sphere.*

4. Equal circular sections are at the same distance from the centre of the sphere, and circular sections at the same distance from the centre are equal. (Cf. III. Prop. 14, parts 1 and 2.)

5. Of two plane sections of a sphere the one which is the nearer to the centre is the greater. (Cf. III. Prop. 15, parts 1 and 2.)

6. Two spheres, which meet one another, must either intersect in a circle whose axis passes through their centres, or touch in a point on the line joining their centres. (Cf. III. Props. 9—13.)

7. Definition. *A straight line and a sphere, which pass through a point but do not intersect there, are said to touch one another at the point.*

*The straight line is called a tangent line to the sphere.*

A tangent line to a sphere is a tangent to the section of the sphere made by any plane passing through the line.

If the straight line joining two points be equal to the sum of the tangents from those two points to a sphere the straight line touches the sphere. (Cf. Ex. 22, p. 274.)

8. All tangent lines to a sphere at a point lie in a plane at right angles to the radius drawn from the point.

Definition. *The plane which contains all the tangent lines at a point is called the tangent plane at the point.*

Spheres are said to intersect at the same angles as their tangent planes.

Two spheres which intersect, intersect at the same angle at all points of their common circle.

9. All tangent lines drawn to a sphere from a given external point are equal, and their points of contact lie on a small circle of which the straight line joining the given point to the centre of the sphere is the axis.

Definition. *The cone generated by the tangent lines drawn from a point is called an enveloping cone of the sphere.*

10. If the point from which the tangent lines are drawn move away from the centre of the sphere, the further it moves, the more nearly are the tangent lines drawn from it parallel to one another, and the more nearly does the circle, on which the points of contact lie, become equal to a great circle.

Hence we say, if the point from which the tangent lines are drawn be at an infinite distance, they lie on a circular cylinder and their points of contact lie on a great circle.

Definition. *A cylinder generated by tangent lines is called an enveloping cylinder of the sphere.*

11. If a straight line be drawn through a given point to cut a given sphere, the intersection of the tangent planes at the two points of section will always lie on a fixed plane. (Cf. III. Add. Prop. on page 258.)

This plane is called the **polar plane** of the given point, and the point is called the **pole** of the plane, with respect to the sphere.

12. A plane through any point cuts a sphere and the polar plane of the point with respect to the sphere in a circle and the polar line of the point with respect to the circle.

If a point and a plane be such that the straight line joining the centre of a sphere to the point is at right angles to the plane, and the rectangle contained by the distances of the point and the plane from the centre is equal to the square on the radius of the sphere, the point is the pole of the plane, and the plane the polar plane of the point with respect to the sphere. (Cf. III. Add. Prop. on page 258.)

If a plane be drawn through a given point to cut a given sphere, the vertex of the enveloping cone which touches the sphere along the section will always lie on the polar plane of the given point. (Cf. III. Add. Prop. on page 258.)

13. If one point lie on the polar plane of another point, the second point lies on the polar plane of the first point.

Thus the pole of any plane through the point  $A$  lies in the polar plane of  $A$ , also the pole of any plane through the point  $B$  lies in the polar plane of  $B$ .

Hence the pole of any plane through the line  $AB$  lies in the common section of the polar planes of  $A$ ,  $B$ .

Similarly the polar plane of any point in the straight line  $AB$  passes through the common section of the polar planes of  $A$  and  $B$ .

14. Two straight lines which are such that the polar plane of any point on either of the lines passes through the other are called **polar lines** of each other with respect to the sphere.

A plane through any line cuts a sphere and the polar line of the given line with respect to the sphere in a circle and the pole of the given line with respect to the circle.

If  $O$  be the centre of the sphere,  $OA$  is at right angles to the polar plane of  $A$  and  $OB$  to the polar plane of  $B$ ; hence the intersection of the polar planes of  $A$  and  $B$  is at right angles to  $OA$  and  $OB$  and therefore to  $AB$ .

Hence two polar lines are at right angles to each other.

15. A chord of a sphere is divided harmonically by any point on it and the polar plane of the point. (Cf. VI. Add. Prop. 7.)

16. All points from which the tangents to two spheres are equal lie on a plane, which is called the **radical plane** of the two spheres. (Cf. III. Add. Prop. on p. 264.)

The section made by any plane of two spheres and their radical plane is two circles and their radical axis.

17. All points from which the tangents to three spheres are equal lie on a straight line which is at right angles to the plane drawn through the centres of the spheres; this is called the **radical axis** of the three spheres. (Cf. III. Add. Prop. on page 264.)

The section made by any plane of three spheres and their radical axis is in general three circles and their radical centre.

The section of three spheres made by any plane through their radical axis is three coaxial circles of which the given straight line is the radical axis.

18. The radical axes of four spheres taken in threes intersect in a point, which is called the **radical centre** of the four spheres. (Cf. Ex. 4 on page 265.)

19. The locus of a point the ratio of whose distances from two given points is constant is a sphere. (Cf. VI. Add. Prop. 5.)

The locus of a point the ratios of whose distances from three given points are constant is a circle.

There are two points whose distances from four given points are in given ratios.

20. *If two solid figures be such that every straight line joining corresponding points of the two figures passes through a fixed point and is divided in a constant ratio at that point, the point is called a centre of similitude of the two figures.*

*If a pair of corresponding points lie on the same side of the centre of similitude, it is called a centre of direct similitude.*

*If a pair of corresponding points lie on opposite sides of the centre of similitude it is called a centre of inverse similitude.*

It should be noticed that two figures which have a centre of direct similitude are similar to the same figure and to one another; but two figures which have a centre of inverse similitude are similar to a pair of reverse figures and as a rule, are not similar to one another.

21. A centre of similitude of two solid figures is also a centre of similitude of their sections made by any plane through the point. (See page 448.)

22. The points, which divide the distance between straight lines joining the centres of two spheres internally and externally in the ratio of their radii, are centres of similitude for the two spheres. (See page 448.)

Every straight line, which passes through the extremities of two parallel radii of two fixed spheres, passes through a centre of similitude of the spheres. (Cf. VI. Add. Prop. 23.)

23. If a sphere be drawn to touch two given spheres, the straight line which passes through the points of contact passes through one of the centres of similitude of the given spheres. (Cf. VI. Add. Prop. 24.)

24. The six centres of similitude of three given spheres lie three by three on four straight lines that lie in the plane which passes through the centres of the spheres. (Cf. VI. Add. Prop. 27.)

*These straight lines are called axes of similitude of the three spheres.*

25. If a sphere be drawn to touch three given spheres the plane drawn through the points of contact passes through one of the axes of similitude. (Cf. VI. Add. Prop. 24.)

26. Two planes can be drawn through each axis of similitude to touch the three given spheres. (Cf. note on page 451.)

Each axis of similitude of three spheres is also an axis of similitude of their sections made by any plane passing through the axis.

27. The twelve centres of similitude of four given spheres taken in pairs, one of the centres of similitude of each pair being taken, lie six by six on eight planes.

These planes are called **planes of similitude of the four spheres.** (Cf. VI. Add. Prop. 27.)

| The centres of similitude which divide |      |      |      |      |      | lie on a plane which may be called |
|--|------|------|------|------|------|------------------------------------|
| $AB$                                   | $AC$ | $AD$ | $CD$ | $BD$ | $BC$ |                                    |
| Externally                             | Ext. | Ext. | Ext. | Ext. | Ext. | $E$                                |
| Internally                             | Int. | Int. | Ext. | Ext. | Ext. | $I_A$                              |
| Internally                             | Ext. | Ext. | Ext. | Int. | Int. | $I_B$                              |
| Externally                             | Int. | Ext. | Int. | Ext. | Int. | $I_C$                              |
| Externally                             | Ext. | Int. | Int. | Int. | Ext. | $I_D$                              |
| Externally                             | Int. | Int. | Ext. | Int. | Int. | $E_{AB}$                           |
| Internally                             | Ext. | Int. | Int. | Ext. | Int. | $E_{AC}$                           |
| Internally                             | Int. | Ext. | Int. | Int. | Ext. | $E_{AD}$                           |

28. If a point on the radical plane of two given spheres be joined to the points of contact of a sphere, which touches both the given spheres, by straight lines which cut the spheres again, another sphere can be described to touch the given spheres at the points of section. (Cf. VI. Add. Prop. 28.)

Hence, by considering the intersection of two of the radical planes of these spheres, we see that: If a point on the radical axis of three given spheres be joined to the points of contact of a sphere, which touches each of the given spheres, by straight lines which cut the

spheres again, another sphere can be described to touch the given spheres at the points of section.

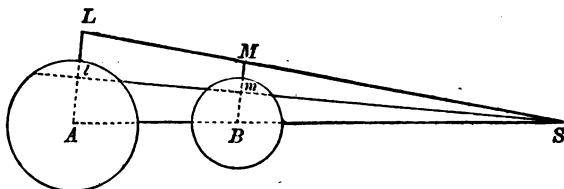
29. If two spheres be drawn to touch four given spheres so that the straight lines joining the points of contact on each of the given spheres passes through the radical centre of the given spheres, the radical plane of the pair of spheres is one of the planes of similitude of the four given spheres. (Cf. VI. Add. Prop. 29.)

30. If a pair of chords of two given spheres intersect in the radical plane and if the chords pass through the poles with respect to the spheres of a plane through one of the centres of similitude, then the points of intersection of the chords with the spheres lie two by two on two straight lines through the centre of similitude. (Cf. VI. Add. Prop. 30.)

31. The poles with respect to two spheres of any plane passing through one of the centres of similitude lie on a straight line through the centre of similitude.

Let  $A, B$  be the centres of the two spheres, whose radii are  $a, b$ , and let  $S$  be one of their centres of similitude.

If  $Al, Bm$  be perpendiculars on the plane  $STU$ , and points  $L, M$  be taken in them such that  $AL \cdot Al = a^2$  and  $BM \cdot Bm = b^2$ , then  $L, M$  are the poles of the plane  $STU$ . (§ 12.)



$$\text{Then } AL \cdot Al : BM \cdot Bm = a^2 : b^2$$

$$= AS^2 : BS^2$$

$$= AS \cdot Al : BS \cdot Bm.$$

$$\text{Therefore } AL : BM = AS : BS.$$

But  $A, B, L, M, S$  all lie in a plane and  $AL, BM$  are parallel;

therefore  $L, M, S$  lie in a straight line.

32. Let  $S$  be one of the centres of similitude of two given spheres whose centres are  $A, B$ .

Let  $L, M$  be the poles of any plane  $STU$  through  $S$  with respect to the spheres  $A, B$  and let two chords through  $L$  and  $M$  meet at  $O$  a point on the radical plane.

Let the plane  $OLM$  cut the spheres in the circles  $PP'p, QQ'q$ .

Because  $LMS$  is a straight line; (§ 31.)

therefore the plane  $OLM$  passes through  $S$ ;

therefore  $L$  and  $M$  are on the polars of  $S$  with respect to the circles  $PP'p, QQ'q$ . (§ 12.)

Again,  $S$  is a centre of similitude of the circles  $PP'p, QQ'q$ . (§ 21.)

And  $O$  is a point on their radical axis. (§ 16.)

therefore if  $OPLp, OQMq$  be straight lines,

$PQ$  passes through  $S$ . (Cf. VI. Add. Prop. 30.)

33. We are now in a position to solve the problem:

*To describe a sphere to touch four given spheres.*

Let  $A, B, C, D$  be four given spheres.

CONSTRUCTION. Find  $O$  the radical centre of the spheres  $A, B, C, D$ , and draw  $E$  one of their planes of similitude, passing through  $X, Y, Z$  centres of similitude of the pairs  $B, C$ ;  $C, A$ ;  $A, B$  respectively.

Find the poles  $K, L, M, N$  of the plane  $E$  with respect to the spheres  $A, B, C, D$  respectively.

Draw  $OK, OL, OM, ON$  and let them produced if necessary cut the spheres  $A, B, C, D$  respectively in  $P, p$ ;  $Q, q$ ;  $R, r$ ;  $S, s$ .

Then the spheres through  $PQRS, pqrs$  will touch the four given spheres.

PROOF. The straight line  $PQ$  passes through  $Z$ , a centre of similitude of the spheres  $A$  and  $B$ . (§ 32.)

Therefore  $Z$  is a centre of similitude of the circular sections of the spheres  $A, B$  made by the plane  $PQR$ . (§ 21.)

Similarly  $QR$  and  $RP$  pass through  $X, Y$  centres of similitude of the spheres  $B, C$  and  $C, A$  respectively.

Hence the circle  $PQR$  touches the sections of the spheres  $A, B, C$  made by the plane  $PQR$ . (Cf. VI. Add. Prop. 31.)

Similarly for the sections made by the planes  $QRS, RSP, SPQ$ .



Again, because the sections of the spheres  $PQRS$  and  $A$  made by two planes touch one another at the point  $P$ , the spheres have a common tangent plane at the point  $P$ , that is the sphere  $PQRS$  touches the sphere  $A$  at the point  $P$ .

Similarly it touches the spheres  $B, C, D$  at the points  $Q, R, S$ .

Therefore the sphere  $PQRS$  touches the four given spheres.

Similarly it may be proved that the sphere  $pqrs$  touches the four given spheres.

34. Since there are eight planes of similitude, and since two spheres can be obtained by the foregoing construction from each plane of similitude, there are  $2 \times 8$  or 16 spheres, which can be described to touch four given spheres.

35. If four spheres touch a fifth sphere, there are three distinct possible types of configuration of the four spheres relatively to the sphere which they touch:

- (1) the four spheres may all lie on the same side of the fifth sphere,
- (2) three of the four spheres may lie on one side and the fourth on the other side of the fifth sphere,
- (3) two of the four spheres may lie on one side and the other two on the other side of the fifth sphere.

Each plane of similitude gives rise to two spheres touching the four given spheres for one of which the nature of the contacts is the same as the nature of the points of similitude in that plane and for the other the reverse.

## INVERSION.

1. In analogy with the definitions used in Plane Geometry (see page 460) the following definitions are used in Solid Geometry.

If  $O$  be a fixed point and  $P$  any other point and if on the straight line  $OP$  (produced if necessary) we take a point  $P'$  such that  $OP \cdot OP' = a^2$  where  $a$  is a constant, then each of the points  $P, P'$  is called the inverse of the other with respect to the sphere whose centre is  $O$  and radius  $a$ .

The straight line  $OP$  is called the radius vector of the point  $P$ .

The point  $O$  is called the pole of inversion and  $a$  the radius of inversion.

If the locus of a point  $P$  be a surface or a curve the locus of the inverse point  $P'$  will be a surface or a curve, which is called the inverse of the locus of  $P$ .

2. It follows that:

*The sections of two inverse surfaces made by any plane through the pole are inverse curves.*

This consideration enables us to state the following theorems:

3. A plane through the pole inverts into itself.

4. A plane which does not pass through the pole inverts into a sphere which passes through the pole and touches there a plane parallel to the given plane.

5. A sphere which passes through the pole inverts into a plane parallel to the tangent plane at the pole.

6. A sphere which does not pass through the pole inverts into a sphere which does not pass through the pole and is such that the pole is a centre of similitude of the two spheres.

7. Two spheres in contact invert into two spheres in contact.

8. Two spheres intersecting at any angle invert into two spheres intersecting at the same angle.

9. A circle inverts into a circle.

This is seen from the facts that every circle may be regarded as the intersection of a pair of spheres, and a pair of spheres inverts into a pair of spheres.

Two circles which are inverses of each other are sections of the same circular cone, whose vertex is the pole of inversion.

10. A system of concentric spheres inverts into a system of coplanar spheres.

11. A system of coplanar spheres inverts into another system of coplanar spheres, and the limiting points into the limiting points.

12. A sphere can be inverted into itself with respect to any point as pole of inversion, if the tangent to the sphere from the point be taken as the radius of inversion.

13. Any two spheres can be inverted into themselves with respect to any point on their radical plane as the pole of inversion.

Any three spheres can be inverted into themselves with respect to any point on their radical axis as the pole of inversion.

Any four spheres can be inverted into themselves with respect to their radical centre as the pole of inversion.

14. Two given spheres can be inverted into two equal spheres with respect to any point on the sphere whose centre is a centre of similitude with respect to the given spheres and whose radius is a mean proportional between the tangents drawn from that centre of similitude to the given spheres.

The sphere on which the centre of inversion lies inverts into the radical plane of the two equal spheres.

15. Three given spheres can be inverted into three equal spheres with respect to any point on certain determinate circles.

The circle on which the centre of inversion is taken inverts into the radical axis of the three equal spheres, that is into the axis of the circle which passes through their centres.

16. Four given spheres can be inverted into four equal spheres with respect to any one of a certain number of points which are given in pairs by the intersection of three spheres.

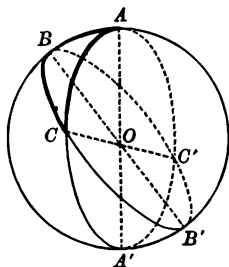
The other point of intersection of the same three spheres inverts into the radical centre of the four equal spheres.

## ELEMENTS OF SPHERICAL TRIGONOMETRY.

**Definition.** *Each of the four parts into which the surface of a sphere is divided by two great circles is called a lune.*

*The angle at which the bounding semicircles intersect is called the angle of the lune.*

It can be shewn by the method of superposition that on equal spheres lunes of equal angles are equal; and hence that the areas of two lunes of different angles are in the same ratio as the angles of the lunes. Hence the area of a lune of angle  $A$  is to the area of the sphere in the same ratio as  $A$  to four right angles.



Let  $OABC$  be a trihedral angle, and let  $ABC$  lie on a sphere whose centre is  $O$ . The triangle  $ABC$  which is formed by the three arcs  $BC$ ,  $CA$ ,  $AB$  of great circles of the sphere, is called a **spherical triangle**: and as the tangents to the arcs  $AB$ ,  $AC$  at  $A$  are at right angles to  $OA$ , the angle between them, which is called the angle  $A$  of the triangle  $BAC$ , is the inclination of the faces  $OBA$ ,  $OCA$ .

The angles  $BOC$ ,  $COA$ ,  $AOB$ , which are in the same ratio as the arcs  $BC$ ,  $CA$ ,  $AB$  of a great circle, are called the **sides** of the spherical triangle  $ABC$  and are denominated  $a$ ,  $b$ ,  $c$  respectively.

We can therefore at once translate the theorems which were established with reference to the plane angles and the inclinations of the faces of a trihedral angle into theorems with reference to the sides and the angles of a spherical triangle.

Thus we may state that in a spherical triangle  $ABC$  whose angles are  $A, B, C$  and whose sides are  $a, b, c$ ,

- (1) If  $a$  is equal to  $b$ ,  $A$  is equal to  $B$ . (Ex. 3, p. 527.)
- (2) If  $A$  is equal to  $B$ ,  $a$  is equal to  $b$ .
- (3) If  $a$  is greater than  $b$ ,  $A$  is greater than  $B$ . (Ex. 4, p. 527.)
- (4) If  $A$  is greater than  $B$ ,  $a$  is greater than  $b$ .
- (5) The sum of any two of the sides  $a, b, c$  is greater than the third side. (Prop. 20.)
- (6) The sum of the sides  $a, b, c$  is less than four right angles. (Prop. 21, Part 1.)

If a sphere whose centre is  $O$  cut the faces of a trihedral angle  $OABC$  in the spherical triangle  $ABC$ , and also cut the faces produced beyond  $O$  in the spherical triangle  $A'B'C'$ , the spherical triangles  $ABC, A'B'C'$  have their corresponding sides and their corresponding angles equal in pairs but it is not possible to shift either of the triangles so as to coincide with the other: two spherical triangles whose sides and angles are equal in pairs but which cannot be made to coincide are called *reverse*.

Since the area of a lune of angle  $A$  is to the area of the sphere in the same ratio as  $A$  to four right angles; therefore if the ratio of  $a$  to 1 be equal to the ratio of  $A$  to four right angles the area of the lune of angle  $A$ : the surface of the sphere  $= a : 1$ , or the area of the lune of angle  $A = aS$ , where  $S$  represents the area of the surface of the sphere. Now take a spherical triangle  $ABC$  and let the ratios of the angles  $A, B, C$  to four right angles be  $\alpha$  to 1,  $\beta$  to 1,  $\gamma$  to 1 respectively; let the great circles of which the sides of the triangle form parts be completed, they will divide the surface of the sphere into four pairs of reverse triangles. Let the reverse triangle of  $ABC$  be  $A'B'C'$ .

Then

the lune  $ABA'C = \alpha S$ ;

the lune  $BCB'A = \beta S$ ;

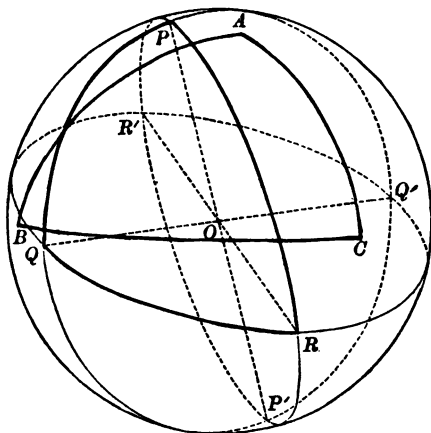
the lune  $CAC'B = \gamma S$ ;

and the area of the triangle  $BAC'$  is equal to the area of the reverse triangle  $CA'B'$ , (Add. Prop. 17.)

therefore the sum of these three lunes is equal to half the surface of the sphere together with twice the area of the triangle  $ABC$ .

Hence,  $\frac{1}{2}S + \text{twice the area of the triangle } ABC = (\alpha + \beta + \gamma) S$ .  
Or the area of the triangle  $ABC = \frac{1}{2}(\alpha + \beta + \gamma - \frac{1}{2}) S$ .

Let  $ABC$  be a spherical triangle; let the axis of the great circle  $BC$  cut the sphere in  $P$  and  $P'$ , in Spherical Trigonometry  $P$  and  $P'$



are called the poles of  $BC$ . Let  $P$  be the pole which lies on the same side of  $BC$  as  $A$ , which is equivalent to saying the arc  $PA$  is less than a quadrant; similarly let  $Q$  and  $R$  be the poles of the great circles  $CA$ ,  $AB$  which lie on the same side of them as  $B$ ,  $C$ , respectively. Then the spherical triangle whose vertices are  $P$ ,  $Q$ ,  $R$  is called the **POLAR TRIANGLE** of  $ABC$ .

$OQ$  and  $OR$  are at right angles to the planes  $OCA$ ,  $OAB$ , therefore the angle  $QOR$  is the supplement of  $A$  the inclination of the planes  $OCA$ ,  $OAB$ . Similarly the angles  $ROP$ ,  $POQ$  are supplements of the angles  $B$  and  $C$ .

Now because  $Q$  is the pole of  $CA$ , the arc  $AQ$  is a quadrant, and because  $R$  is the pole of  $AB$ , the arc  $AR$  is a quadrant, and because the arcs  $AQ$  and  $AR$  are quadrants,  $A$  is a pole of  $QR$ .  $A$  is on the same side of  $QR$  as  $P$ , because the arc  $AP$  is less than a quadrant. Similarly  $B$ ,  $C$  are poles of  $RP$  and  $PQ$  and on the same side of them as  $Q$  and  $R$  respectively.

Hence if  $PQR$  be the polar triangle of  $ABC$ , then  $ABC$  is the polar triangle of  $PQR$ , and the sides of each are the supplements of the angles of the other.

In the polar triangle  $PQR$  any two sides are together greater than the third, i.e.  $q + r$  is greater than  $p$ , therefore in the original triangle  $ABC$ , the sum of the supplements of  $B$  and  $C$  is greater than the supplement of  $A$ ; or,  $A + \text{two right angles}$  is greater than  $B + C$ ; therefore in a spherical triangle any one of the three angles together with two right angles is greater than the sum of the other two angles.

Now in the polar triangle  $PQR$ , the sum of the sides  $p, q, r$  is less than four right angles (Prop. 21, Part 1); therefore in the triangle  $ABC$ , the sum of the supplements of the angles  $A, B, C$  is less than four right angles; or  $A + B + C$ , that is the sum of the three angles of a spherical triangle, is greater than two right angles.

Two triangles  $ABC, A'B'C'$  on the surface of the same sphere are either equal in all respects or reverse

(I) If  $a = a', b = b'$  and  $C = C'$ . (Add. Prop. 2.)

(II) If  $a = a', b = b'$  and  $c = c'$ . (Add. Prop. 1.)

(III) If  $a = a', b = b'$  and  $A = A'$  and if in addition

(1)  $B$  and  $B'$  be each less than a right angle,

or (2)  $B$  and  $B'$  be each greater than a right angle,

or (3) either  $B$  or  $B'$  be a right angle. (Add. Prop. 3.)

And hence in consequence of the properties of the polar triangle,

(IV) If  $A = A', B = B'$  and  $c = c'$ . (Add. Prop. 2.)

(V) If  $A = A', B = B'$  and  $C = C'$ . (Add. Prop. 1.)

(VI) If  $A = A', B = B'$  and  $a = a'$  and if in addition

(1)  $b$  and  $b'$  be each greater than a right angle,

or (2)  $b$  and  $b'$  be each less than a right angle,

or (3) either  $b$  or  $b'$  be a right angle. (Add. Prop. 3.)

It will be observed that the geometry of great circles on the surface of a sphere differs in several important particulars from the geometry of straight lines in a plane.

Whereas, in a plane triangle, the sum of the angles is always equal to two right angles, in a spherical triangle, the sum of the angles is always greater than two right angles.

Whereas in a plane triangle, when two angles are given, the third angle and the shape but not the size of the triangle is determinate; in a spherical triangle when two angles are given the third angle and the shape and the size of the triangle are all indeterminate, but three angles determine both the shape and the size of the triangle.

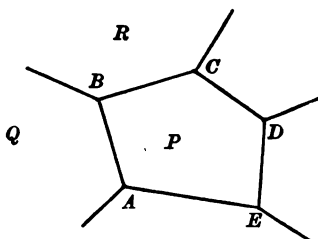
Again, whereas in a plane triangle all equilateral triangles are similar and have angles of the same size, the angles in an equilateral spherical triangle may have any value between two thirds of a right angle and two right angles, the smaller value occurring when the triangle is extremely small, and the larger value, when the triangle approximates to a great circle.



RECIPROCAL FIGURES.

If we consider any polyhedron, and take the pole of each face with respect to a sphere;

then if  $P, Q, R...$  be a number of faces meeting in a point  $B$ , and  $p, q, r...$  be their poles, the points  $p, q, r...$  will lie in a plane which is the polar plane of  $B$ , and  $pq$  is the polar line of  $AB$ , the intersection of the planes  $P$  and  $Q$ .



Therefore corresponding to an  $n$ -sided face  $P$  there exists a point  $p$ , in which  $n$  lines  $pq, pr, ps...$  meet,

and to an  $m$ -faced corner  $B$ , corresponds an  $m$ -sided face  $pqr...$

It appears therefore that it is possible to construct a new polyhedron having its corners, faces and edges equal in number to the faces, corners and edges of the original polyhedron, such that to every face of  $n$  sides in one corresponds a corner of  $n$  faces in the other, and vice versa, and to every edge of one corresponds an edge of the other.

Two figures so related are called **reciprocal figures**.



The perpendiculars from any point in the plane  $LBC$  on the planes  $DBC$ ,  $ABC$  are equal; similarly the perpendiculars from any point in the plane  $MCA$  upon the two planes  $DCA$ ,  $BCA$  are equal, and also the perpendiculars from any point in the plane  $NAB$  upon the planes  $DAB$ ,  $CAB$  are equal.

The three planes  $LBC$ ,  $MCA$ ,  $NAB$  will intersect in some point  $I$ .  
(Prop. 3.)

The perpendiculars from the point  $I$  on the four planes  $BCD$ ,  $CDA$ ,  $DAB$ ,  $ABC$  are equal, and a sphere whose centre is  $I$  and whose radius is equal to one of these perpendiculars will pass through the extremities of all four and touch the planes  $BCD$ ,  $CDA$ ,  $DAB$ ,  $ABC$ , and will be inscribed in the tetrahedron  $ABCD$ .

It may be proved in the same way that, if the three dihedral angles along the edges  $BC$ ,  $CA$ ,  $AB$  be bisected either internally or externally by planes  $LBC$ ,  $MCA$ ,  $NAB$ , then the point  $I$ , where these planes meet is the centre of a sphere which touches the four faces of the tetrahedron.

Because the perpendiculars from the point  $I$  on the four faces of the tetrahedron are equal it follows that one of the planes, which bisect the dihedral angles of the tetrahedron along each of the remaining edges  $DA$ ,  $DB$ ,  $DC$ , internally or externally, passes through the point  $I$ , in other words each point  $I$  lies on one or other of the planes bisecting, externally or internally, the dihedral angle of a tetrahedron along each of its six edges.

As each of the bisecting planes  $LBC$ ,  $MCA$ ,  $NAB$  can be chosen in either of two ways, it follows that the number of positions of the point  $I$ , or what is the same thing the number of spheres which touch the four faces of the tetrahedron, is  $2 \times 2 \times 2 = 8$ .

The geometrical relations between the centres of the eight spheres and the angular points and the edges of the tetrahedron are more fully exhibited in the Table which is to be found on page 613.

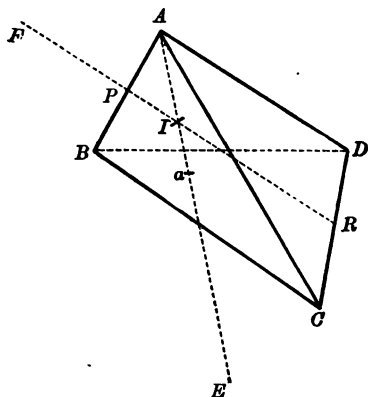
In general each set of three bisecting planes meets in a point at a finite distance from the tetrahedron; but in certain cases the three planes are parallel to the same line, and the centre is then at an infinite distance.

Now if  $I$  be the centre of the inscribed sphere and  $AI$  produced meet the face  $BCD$  in  $a$ ,

then  $Aa$  is to  $Ia$  as the tetrahedron  $ABCD$  is to the tetrahedron  $IBCD$ :

therefore  $AI$  is to  $Ia$  as the sum of the tetrahedrons  $IACD$ ,  $IADB$  and  $IABC$  to the tetrahedron  $IBCD$ , that is, as  $b+c+d$  to  $a$ ,

where for brevity  $a, b, c, d$ , are used to represent the areas of the faces of the tetrahedron  $ABCD$  opposite to  $A, B, C, D$ , respectively.



Let  $AI$  produced meet the plane which bisects externally the dihedral angle along the edge  $BC$  at  $E$ .

**$E$  is the centre of the escribed sphere beyond the face  $DCB$ .**

*Aa* is to *aE* as the tetrahedron *ABCD* to the tetrahedron *EBCD* :

therefore  $AE$  is to  $Ea$  as the sum of the tetrahedrons  $EACD$ ,  $EADB$  and  $EABC$  to the tetrahedron  $EBCD$ , that is, as  $b+c+d$  to  $a$ ;

therefore  $AI$  is to  $Ia$  as  $AE$  to  $Ea$ .

therefore  $AIaE$  is a Harmonic range.

Let the internal bisector of the dihedral angle along the edge  $AB$   
cut the edge  $CD$  in  $R$ ,  
and let the internal bisector of the dihedral angle along the edge  $CD$   
cut the edge  $AB$  in  $P$ .

Then  $I$  the centre of the inscribed sphere must lie in  $PR$ .  
 $IP$  is to  $PR$  as the tetrahedron  $IABD$  is to the tetrahedron  $RABD$ ,  
and also as the tetrahedron  $IABC$  is to the tetrahedron  $RABC$ ;  
therefore as the sum of the tetrahedrons  $IABD$  and  $IABC$  to the sum  
of the tetrahedrons  $RABD$  and  $RABC$ ,  
that is, as the sum of the tetrahedrons  $IABD$  and  $IABC$  to the tetra-  
hedron  $ABCD$ ,

that is, as  $c+d$  to  $a+b+c+d$ .

Therefore  $PI$  is to  $IR$  as  $c+d$  is to  $a+b$ .

The plane which bisects the dihedral angle along the edge  $AC$   
externally will meet  $PR$  produced; it must meet  $PR$  produced either  
beyond  $P$  or beyond  $R$ , let it meet beyond  $P$  in  $F$ .

Then  $FP$  is to  $PR$  as the tetrahedron  $FABD$  is to the tetrahedron  
 $RABD$ ,

and as the tetrahedron  $FABC$  is to the tetrahedron  $RABC$ ,  
and therefore as the sum of the tetrahedrons  $FABD$  and  $FABC$  to  
the tetrahedron  $ABCD$ ;

therefore  $FP$  is to  $FR$  as the sum of the tetrahedrons  $FABD$  and  
 $FABC$  to the sum of the tetrahedrons  $FBCD$  and  $FACD$ ,

that is, as  $c+d$  to  $a+b$ .

This proves that the point  $F$  lies outside the edge  $AB$ , if  $a+b$  be  
greater than  $c+d$ , and outside the edge  $CD$  if less.

If  $a+b=c+d$ ,  $F$  is at an infinite distance in  $PR$ , or, in other  
words, the plane which bisects the dihedral angle along the edge  $AC$   
externally is parallel to the line  $PR$ .

We also see that

$FP$  is to  $FR$  as  $IP$  to  $IR$ ;

that is  $FPIR$  is a Harmonic range.

Again, using  $a, b, c, d$  to represent the areas of the faces of the tetrahedron  $ABCD$  opposite  $A, B, C, D$  respectively,

if  $a, b, c, d$  be in descending order of magnitude, it follows as a matter of course

that  $a + b$  is greater than  $c + d$ ,

and that  $a + c$  is greater than  $b + d$ ;

but  $a + d$  might be either greater than, equal to, or less than  $b + c$ .

Take first the case where  $a + d$  is greater than  $b + c$ ; from what has gone before, it follows that in this case the tetrahedron  $ABCD$  has escribed spheres beyond  $AB, AC, AD$ , three edges of the tetrahedron meeting in a point.

Next let us take the case where  $a + d$  is less than  $b + c$ ; in this case it follows from what has gone before that the tetrahedron has escribed spheres beyond  $AB, AC$ , and  $BC$ , three edges of the tetrahedron lying in a face.

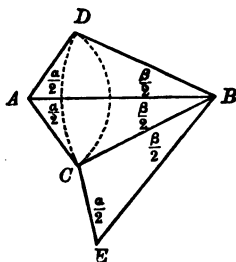
Thus we have two distinct types of tetrahedron. If  $ABCD$  were a regular tetrahedron and  $G$  and  $H$  were any two points in the perpendicular from  $A$  on the face  $BCD$ , one on either side of  $A$ , then the tetrahedrons  $GBCD, HBCD$  belong one to each of these two types of tetrahedron; but the regular tetrahedron  $ABCD$  does not possess a finite sphere escribed beyond any edge.

In the following Table the centre of the inscribed sphere is denoted by  $I$ ; the centre of the sphere escribed outside the face  $BCD$  by  $E_A$ , this sphere touches the face  $BCD$  in a point within the triangle  $BCD$ , and touches the faces  $ABC, ACD, ADB$  in points in the faces produced beyond  $BC, CD, DB$  respectively; similarly the centres of the spheres escribed outside the faces  $CDA, DAB, ABC$  are denoted by  $E_B, E_C, E_D$  respectively. The centre of the sphere which is escribed outside the edge  $AB$  or  $CD$  is denoted by  $I_{AB}$ , this sphere if escribed beyond  $AB$  touches the faces  $ABC, ABD$  each produced beyond  $AB$ , the face  $ACD$  produced beyond  $A$ , the face  $BCD$  produced beyond  $B$ ; similarly the centres of the sphere escribed outside  $AC$  or  $BD$ , and outside  $AD$  or  $BC$  are denoted by  $I_{AC}$  and  $I_{AD}$  respectively.

| The plane bisecting the dihedral angle along the edge |           |           |           |           |           | passes through        |
|---|-----------|-----------|-----------|-----------|-----------|-----------------------|
| <i>AB</i>   | <i>AC</i> | <i>AD</i> | <i>CD</i> | <i>BD</i> | <i>BC</i> |                       |
| Internally  | Int.      | Int.      | Int.      | Int.      | Int.      | <i>I</i>              |
| Internally  | Int.      | Int.      | Ext.      | Ext.      | Ext.      | <i>E<sub>A</sub></i>  |
| Internally  | Ext.      | Ext.      | Ext.      | Int.      | Int.      | <i>E<sub>B</sub></i>  |
| Externally  | Int.      | Ext.      | Int.      | Ext.      | Int.      | <i>E<sub>C</sub></i>  |
| Externally  | Ext.      | Int.      | Int.      | Int.      | Ext.      | <i>E<sub>D</sub></i>  |
| Internally  | Ext.      | Ext.      | Int.      | Ext.      | Ext.      | <i>I<sub>AB</sub></i> |
| Externally  | Int.      | Ext.      | Ext.      | Int.      | Ext.      | <i>I<sub>AC</sub></i> |
| Externally  | Ext.      | Int.      | Ext.      | Ext.      | Int.      | <i>I<sub>AD</sub></i> |

## ROTATION ABOUT A POINT IN A PLANE.

1. Any plane figure can by rotation round some point in the plane through a definite angle, be shifted from any one position to any other position, which it can occupy in the same plane without being turned over. (See pages 186, 187 and Exercises 3, 4.) It follows that the change in position produced by successive rotations round two or more given points, in a given order, through given angles, can be produced by a single rotation round a definite point through a definite angle. If a figure rotate round two points,  $A$  and  $B$ , in order through angles  $\alpha$  and  $\beta$ , in the same sense, the change in position is the same as would be produced by a rotation through an angle  $\alpha + \beta$  round a point  $C$ , such that the angles  $CAB$ ,  $ABC$ , are  $\frac{1}{2}\alpha$  and  $\frac{1}{2}\beta$ , where the angle  $BAC$  is drawn in the opposite sense to the rotation round  $A$ , and the angle  $ABC$  is drawn in the same sense as the rotation round  $B$ .



For in the diagram rotation round  $A$  through an angle  $\alpha$  would transfer a point from  $C$  to the position  $D$ , subsequent rotation round  $B$  in the same sense through an angle  $\beta$  would transfer the point back again from  $D$  to its original position  $C$ .

$C$  therefore must be the point round which the equivalent rotation takes place.

2. Again, the rotation round  $A$  shifts a line from the position  $CA$  to the position  $DA$ , and the subsequent rotation round  $B$  shifts the line from the position  $DA$  to the position  $CE$ .

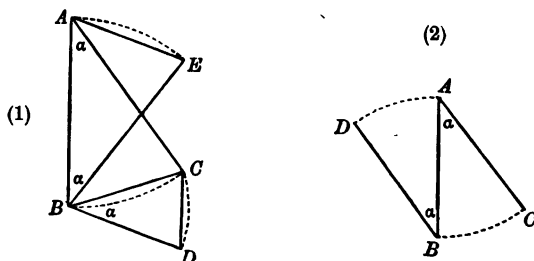
The equivalent rotation therefore shifts a line from the position  $CA$  to the position  $CE$ , that is to say it rotates it through the angle  $\angle ACE$ , which is equal to  $\alpha + \beta$ .



If the rotations  $\alpha$  and  $\beta$  round  $A$  and  $B$  were in opposite senses, the equivalent rotation round  $C$  would be  $\alpha - \beta$ .

3. If the rotations  $\alpha$ ,  $\beta$  be equal and in opposite senses, the lines  $AC$ ,  $BC$  in the diagram would be parallel, that is, the point  $C$  would be at an infinite distance. This consideration suggests that the result of two equal rotations in opposite senses is to shift every point of a figure through the same distance in a fixed direction.

To prove that such is the case we will consider the effect produced on a straight line  $AB$  by successive rotations through an angle  $\alpha$  in opposite senses round  $A$  and  $B$ .



In the accompanying diagram (1) a rotation round  $A$  through an angle  $\alpha$  will leave a point at  $A$  unchanged, but will shift a point from  $B$  to  $C$ . A subsequent rotation round  $B$  through an angle  $\alpha$  in the opposite sense will shift a point from  $A$  to  $E$  and a point from  $C$  to  $D$ .

The three triangles  $BAC$ ,  $CBD$ ,  $ABE$  are similar, it follows that  $AE$  is equal and parallel to  $BD$ ; and every other point in the figure must move through a distance equal and parallel to  $AE$ .

A change of position in which every point in the figure is shifted through the same distance parallel to a fixed straight line is called a **translation**.

4. Rotation round a point through a given angle is equivalent to a rotation round any other point through the same angle in the same sense followed by a definite translation.

For in the accompanying diagram (2) rotation round  $A$  through an angle  $\alpha$  will shift a line from the position  $AB$  to the position  $AC$ . An equal rotation round  $B$  in the same sense would shift the line from the position  $BA$  to the position  $BD$ . And the line will be shifted from either of the positions  $AC$ ,  $DB$  to the other by a simple translation.

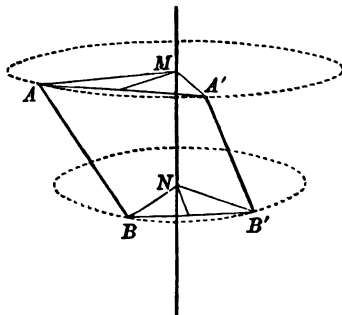
## ROTATION ABOUT A LINE IN SPACE.

5. *Any finite straight line may be shifted from any one position to any other by rotation round some straight line.*

Let  $AB$ ,  $A'B'$  be any two positions of a finite straight line.

Draw the plane  $HMN$  which bisects  $AA'$  at right angles, and also the plane  $KMN$  which bisects  $BB'$  at right angles;

Let  $MN$  the common section of these planes meet the planes drawn through  $AA'$ ,  $BB'$  at right angles to it in  $M$ ,  $N$  respectively.



The tetrahedrons  $AMNB$ ,  $A'MNB'$  are equal in all respects.

And  $MN$  is fixed.

Therefore the line can be shifted from the position  $AB$  to the position  $A'B'$  by rotation round  $MN$ .

6. If the lines  $AA'$ ,  $BB'$  be parallel, the two planes which bisect them at right angles are parallel: in other words, the axis of rotation in this special case is at an infinite distance, or what is equivalent the shift from the position  $AB$  to  $A'B'$  is obtained by a simple translation.

7. If the two planes  $HMN$ ,  $KMN$  be not only parallel but coincident their common section becomes indeterminate and any straight line  $MN$  in their plane, which passes through the intersection of  $AB$ ,  $A'B'$ , may be taken as an axis of rotation.

8. Because the tetrahedrons  $AMNB$ ,  $A'MNB'$  are equal in all respects, therefore the dihedral angles between the pairs of planes  $MNA$ ,  $MNB$  and  $MNA'$ ,  $MNB'$  are equal.

Therefore the dihedral angles between the pairs of planes  $MNA$ ,  $MNA'$  and  $MNB$ ,  $MNB'$  are equal.

Therefore the plane angles  $AMA'$ ,  $BNB'$  are equal (Prop. 7).

Also  $MA=MA'$  and  $NB=NB'$  and  $MN$  is at right angles to the planes  $AMA'$ ,  $BNB'$ .

Hence when a figure rotates round a straight line, every point of the figure rotates round a point of that straight line in a plane at right angles to it through the same angle.

9. The converse of this theorem is also true.

When every point of a figure rotates round a point of a straight line in a plane at right angles to it through the same angle, the figure rotates round the straight line.

10. Therefore the theorems, which we have established for rotation in a plane about points in the plane and translation in the plane, may be extended to rotation about parallel straight lines in space and translation at right angles to those lines.

11. If  $AB$  a straight line in a figure  $S$  have been shifted so that  $AB$  coincides with  $A'B'$  the corresponding straight line in an equal figure  $S'$  and a point  $C$  in the figure  $S$  not collinear with  $AB$  coincides with the corresponding point  $C'$  in the figure  $S'$ , then the figure  $S$  coincides with the figure  $S'$ .

But if when  $AB$  is shifted to  $A'B'$ ,  $C$  do not take the position  $C'$ , we can make it do so by rotation through some angle round the line  $A'B'$ , and if this be done the figure  $S$  will coincide with the figure  $S'$ . It appears therefore that any solid figure can be shifted into any other position in space by successive rotations, round two straight lines.

12. NOTE. If  $ABCD$ ,  $A'B'C'D'$  be two tetrahedrons in which  $AB$ ,  $A'B'$ , are at right angles to the planes  $CBD$ ,  $C'B'D'$ , then if the tetrahedrons be equal the angles  $CBD$ ,  $C'B'D'$ , would be said to be drawn in the same sense round  $AB$ ,  $A'B'$ , but if the tetrahedrons were reverse the angles  $CBD$ ,  $C'B'D'$ , would be said to be drawn in opposite senses round  $AB$ ,  $A'B'$ .

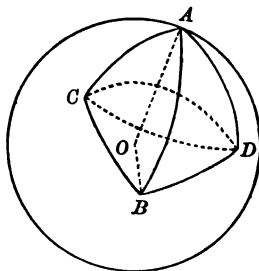
13. We will now prove the theorem: *Successive rotations through given angles round two given intersecting straight lines are equivalent to rotation through a definite angle round a definite straight line which passes through the intersection of the given straight lines.*

Let  $OA$ ,  $OB$  be the axes of rotation and  $\alpha$ ,  $\beta$  the angles of rotation round them.

First let the rotation be in the same sense.

Draw a sphere whose centre is  $O$  to cut  $OA$ ,  $OB$  in  $A$  and  $B$ .

On the surface of this sphere draw the spherical triangles  $ABC$ ,  $ABD$  making the angles  $BAC$ ,  $BAD$  each equal  $\frac{1}{2}\alpha$  and  $ABC$ ,  $ABD$  each equal to  $\frac{1}{2}\beta$ , the angle  $BAC$  being measured in the opposite sense to the rotation round  $OA$ , and the angle  $ABC$  being measured in the same sense as the rotation round  $OB$ .



In the diagram rotation round  $OA$  through an angle  $\alpha$  would transfer a point from  $C$  to the position  $D$ . Subsequent rotation round  $OB$  in the same sense through an angle  $\beta$  would transfer the point back again from  $D$  to its original position  $C$ .

$OC$  must therefore be the axis round which the equivalent rotation takes place.

In the same way as in § 2 it may be shewn that the angle of the equivalent rotation round  $OC$  is equal to twice the supplement of the angle  $ACB$ .

The equivalent rotation is always less than the sum of the rotations  $\alpha$  and  $\beta$  (p. 605).

Secondly, let the rotations about  $OA$  and  $OB$  be in opposite senses. In this case we may consider the rotation  $\beta$  about  $OB$ , as a rotation  $\beta$  about  $OB'$ , the line  $BO$  produced beyond  $O$ , in the same sense as the rotation  $\alpha$  about  $OA$  and thus reduce the second case to the first.

14. We will now prove the following theorem: *Successive rotations through given angles round two given non-intersecting straight lines are equivalent to a rotation through a definite angle round a definite straight line together with a translation parallel to that line.*

Let us consider the case of successive rotations through angles  $\alpha$  and  $\beta$ , round a straight line  $OA$  and a straight line  $PQ$  parallel to  $OB$ .

The rotation round  $PQ$  is equivalent to a rotation round  $OB$  followed by a translation at right angles to  $OB$ . (§§ 4, 10.) And successive rotations round  $OA$  and  $OB$  through angles  $\alpha$  and  $\beta$  are equivalent to a rotation round  $OC$  through a definite angle  $\gamma$  (§ 13).

Therefore the successive rotations round  $OA$  and  $PQ$  are equivalent to a rotation round  $OC$  followed by a translation at right angles to  $OB$ .

Again, the translation at right angles to  $OB$  is equivalent to a definite translation at right angles to  $OC$  followed by a definite translation parallel to  $OC$ ; and the rotation round  $OC$  followed by a translation at right angles to  $OC$  is equivalent to a rotation through the angle  $\gamma$  about some definite straight line  $MN$  parallel to  $OC$ .

Therefore the rotation round  $OC$  followed by the translation at right angles to  $OB$  is equivalent to the rotation round  $MN$ , followed by the translation parallel to  $MN$ .

Here the translation is parallel to the axis of rotation, so that the order in which the translation and rotation take place is immaterial.

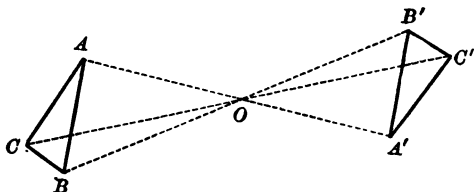
We have already shewn that a figure may be moved from any position to any other by successive rotations round two straight lines.

Hence a figure can be shifted from one given position to another given position by rotation round a definite straight line together with a translation parallel to it.

## REFLECTION IN A POINT.

*Two figures  $S, S'$ , such that every straight line joining a pair of corresponding points  $A, A'$  in the two figures is bisected at a fixed point  $O$ , are called reflexes of each other in the point  $O$ .*

Every line  $AB$  in the figure  $S$  is equal to the corresponding line  $A'B'$  in the figure  $S'$ , and every triangle  $ABC$  in  $S$  is equal to the corresponding triangle  $A'B'C'$  in  $S'$  and, in plane geometry, the triangle  $ABC$  can be shifted into the position  $A'B'C'$  by rotation in its plane through two right angles round the point  $O$ .



Every tetrahedron  $ABCD$  in  $S$  has its edges equal to the edges of the corresponding tetrahedron  $A'B'C'D'$  in  $S'$ , but it is impossible to shift the tetrahedron  $ABCD$  so as to coincide with the tetrahedron  $A'B'C'D'$ . They are reverse tetrahedrons. (Add. Prop. 7, p. 561.)

If  $S''$  be the reflex of  $S'$ , with respect to the point  $O'$ , the figure  $S$  can be shifted to the position  $S''$  by simple translation through a distance equal and parallel to  $2OO'$ .

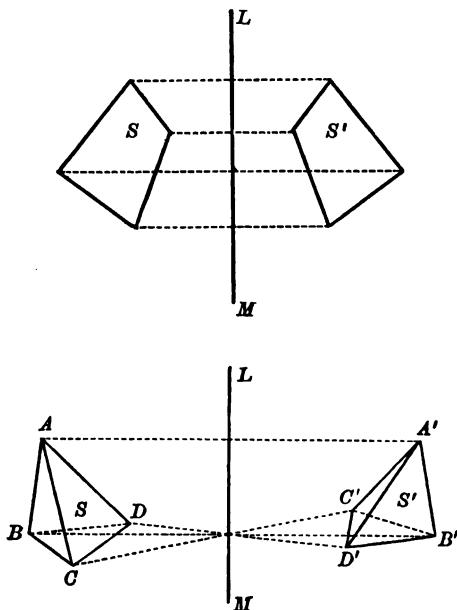
Any even number of successive reflections in points are equivalent to a single translation.

Any odd number of successive reflections in points are equivalent to a single reflection in a point.

## REFLECTION IN A STRAIGHT LINE.

Two figures  $S, S'$ , such that every straight line joining two corresponding points  $A, A'$  in the two figures is bisected at right angles by a fixed straight line  $LM$  are called **reflexes** of each other in the line  $LM$ .

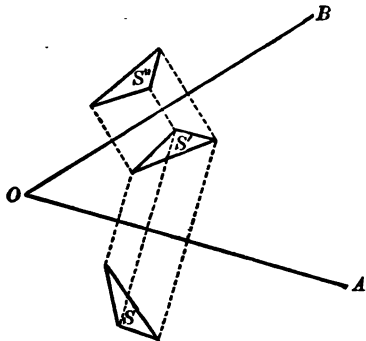
If a figure  $S$  be the reflex of the figure  $S'$  in a straight line  $LM$ , either figure can be shifted into the position of the other by rotation round the line  $LM$  through two right angles (p. 616).



The figure  $S$  can be shifted into the position  $S'$  by rotation through two right angles round the line  $LM$ .

Therefore reflection in a line is equivalent to reflection in a parallel line followed by a translation parallel to a line, that cuts both these lines at right angles (p. 617).

If  $S'$  be the reflex of  $S$  with respect to the line  $OA$  and  $S''$  the reflex of  $S'$  with respect to the line  $OB$ ,  $S$  can be removed to the position  $S''$  by rotation through an angle equal to twice  $\angle O$  round  $OC$ , a line at right angles to the plane  $AOB$ .



This proves that by successive reflections in two straight lines in the plane, a plane figure can be shifted from any position to any other position in its plane, which it can occupy without being turned over.

It can also be proved that

A figure can be changed from any position in space to any other by successive reflections in two straight lines.

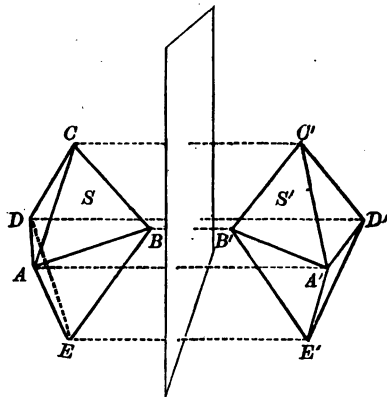
For the figure can be shifted from any position to any other by rotation round some straight line  $OC$  through a definite angle followed by a definite translation parallel to  $OC$  (p. 619); the rotation round  $OC$  can be replaced by successive rotations through two right angles round  $OA$ ,  $OB$  two straight lines at right angles to  $OC$ . The rotation round  $OB$  followed by the translation parallel to  $OC$  is equivalent to a rotation through two right angles round some line parallel to  $OB$ , and intersecting  $OC$ .



## REFLECTION IN A PLANE.

Two figures  $S, S'$ , such that every straight line joining two corresponding points  $A, A'$  in the two figures is bisected at right angles by a fixed plane  $LMN$ , are called **reflexes** of each other in the plane  $LMN$ .

Every line  $AB$  in the figure  $S$  is equal to the corresponding line  $A'B'$  in the figure  $S'$ , and two corresponding tetrahedrons  $ABCD, A'B'C'D'$  are reverse tetrahedrons.



If  $S''$  be the reflex of  $S'$  in the plane  $LMP$ , then  $S$  can be shifted into the position  $S''$  by rotation round  $LM$  through an angle double of the inclination of the planes.

Hence reflection in a line is equivalent to successive reflections in two planes drawn through the line at right angles to each other.

Similarly reflection in a point is equivalent to successive reflections in three planes drawn through the point at right angles to each other.

Successive reflections in two parallel planes are equivalent to a translation at right angles to the planes through a distance equal to double the distance between the planes.

A figure can be shifted from any position to any other position by successive reflections in two intersecting planes together with reflections in two planes at right angles to the first pair of planes.

If a part of a figure be the reflex of the other part with respect to a point, a line or a plane, the figure is said to be **symmetrical** with respect to the point, the line or the plane.

### SPHERE TOUCHING THE FOUR SIDES OF A SKEW QUADRILATERAL.

*To draw a sphere to touch the four sides of a skew quadrilateral.*

Let  $ABCD$  be the quadrilateral.

In the plane  $ABC$  draw  $B\beta$ , one of the bisectors of the angle  $ABC$ .

Through  $B$  draw a plane  $LBb$  at right angles to  $B\beta$ ; every point in this plane is equidistant from  $AB$  and  $BC$ . (Exercise 5, p. 511.)

Similarly draw  $C\gamma$  and  $D\delta$ , bisectors of the angles  $BCD$ ,  $CDA$ , and draw the planes  $MCc$  and  $NDd$  at right angles to them.

The three planes  $LBb$ ,  $MCc$ ,  $NDd$  will meet in some point  $O$ . (Prop. 3.)

Then  $OP$ ,  $OQ$ ,  $OR$ ,  $OS$ , perpendiculars drawn to  $AB$ ,  $BC$ ,  $CD$ ,  $DA$  are all equal; and a sphere drawn with centre  $O$  and radius  $OP$  will touch the four lines at  $P$ ,  $Q$ ,  $R$ ,  $S$ .

Generally the point  $O$  is a unique point at a finite distance from the given lines.

Since there are two planes which can be drawn through each of the three points  $B$ ,  $C$ ,  $D$  at right angles to one of the bisectors of the angle at the point, there are  $2 \times 2 \times 2 = 8$  combinations of three planes, each of which will give the centre of a sphere touching the four sides of the quadrilateral.

There are therefore in general 8 finite solutions of the problem. But if three of the bisectors of the angles at  $B$ ,  $C$ ,  $D$  say  $B\beta$ ,  $C\gamma$ ,  $D\delta$  be parallel to one plane, the three corresponding planes  $LBb$ ,  $MCc$ ,  $NDd$  are at right angles to that plane, and their point of intersection  $O$  will be at an infinite distance. And further if  $B\beta$ ,  $C\gamma$ ,  $D\delta$  be parallel to one plane and in addition the three planes  $LBb$ ,  $MCc$ ,  $NDd$  intersect in a common line  $OO'$ , the point  $O$  may be taken at any

point in the common section of the planes. And thus instead of a single solution we obtain in this case an infinite series of solutions.

If  $op$  be at right angles to  $OO'$  and to  $AB$ ,  $o$  is the centre of the least sphere, that can be drawn in the series, and  $op$  is its radius. (Ex. 4, p. 543.)

If the three lines  $B\beta$ ,  $C\gamma$ ,  $D\delta$  which we have assumed to be parallel to one plane be all external bisectors, the three solutions which are obtained by taking two external and one internal bisector occur among the series, but the four solutions obtained by taking one external and two internal bisectors, or three internal bisectors, still remain distinct from the series.

Draw  $PQ$ ,  $QR$ ,  $RS$ ; they are parallel to the lines  $B\beta$ ,  $C\gamma$ ,  $D\delta$ .

If therefore  $B\beta$ ,  $C\gamma$ ,  $D\delta$  be not parallel to one plane,  $P$ ,  $Q$ ,  $R$ ,  $S$  are not in one plane. But if  $B\beta$ ,  $C\gamma$ ,  $D\delta$  be parallel to one plane,  $P$ ,  $Q$ ,  $R$ ,  $S$  are in one plane and since  $P$ ,  $Q$ ,  $R$ ,  $S$  are points on the surface of a sphere, the quadrilateral  $PQRS$  can be inscribed in a circle.

The line  $PS$  is parallel to one of the bisectors of the angle  $BAD$ .

We conclude that, if one of the bisectors of each of three angles of a skew quadrilateral be parallel to one plane, one of the bisectors of the fourth angle is parallel to the same plane.

If a plane meet the sides of a skew quadrilateral it must meet an even number of sides in points in the sides themselves and an even number in the sides produced (Ex. 5, p. 509).

Let us assume that the four points  $P$ ,  $Q$ ,  $R$ ,  $S$  lie all in the sides.

Then  $AP=AS$ ,  $BP=BQ$ ,  $CR=CQ$ ,  $DR=DS$ .

Therefore  $AB + CD = BC + DA$ .

This therefore is a relation between the sides of the quadrilateral which must exist in order that the points of contact with the sphere,  $P$ ,  $Q$ ,  $R$ ,  $S$ , may lie all in the sides.

Similarly it can be proved in every other case, which can occur, of the four points of contact lying in a plane, that either the sums of opposite sides or the differences of opposite sides of the quadrilateral are equal.

Hence the theorem:—If three of the bisectors of the angles of a skew quadrilateral be parallel to one plane, either the sums of opposite sides or the differences of opposite sides are equal.

The converse of this proposition may be stated thus :—*If in a skew quadrilateral either the sums or the differences of opposite sides be equal, four of the bisectors of the angles are parallel to one plane.*

In the quadrilateral  $ABCD$  let  $AB + CD = AD + BC$ .

Take any point  $P$  in  $AB$  and complete the quadrilateral  $PQRS$ , so that

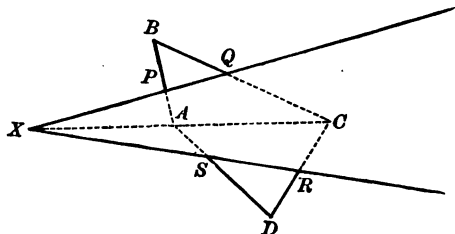
$$BP = BQ, CQ = CR, DR = DS \text{ and } AS = AP.$$

Planes drawn through  $P, Q, R$  at right angles to  $AB, BC, CD$  will meet at some point  $O$ . A sphere can be drawn with centre  $O$  to touch  $AB, BC, CD$  at  $P, Q, R$ .

$AP, DR$  are tangents to this sphere and  $AP + DR = AD$ .

Therefore  $AD$  touches the sphere (§ 7, p. 592). It must touch at  $S$ .

Let  $QP, CA$  produced if necessary meet in  $X$ .



Then

$$\begin{aligned} AX : CX &= \text{perpendicular from } A \text{ on } PQ : \text{perp. from } C \text{ on } PQ \\ &= AP : CQ \\ &= AS : CR \\ &= \text{perpendicular from } A \text{ on } RS : \text{perp. from } C \text{ on } RS. \end{aligned}$$

Therefore  $RSX$  is a straight line.  $PQRS$  is a plane quadrilateral ; and each of its sides is parallel to one of the bisectors of  $A, B, C$  or  $D$ .

A similar proof can be given in the case when the differences of opposite sides of the quadrilateral  $ABCD$  are equal.

# SPHERE TOUCHING STRAIGHT LINES. 627

Let  $O$  be the centre of a sphere whose points of contact are  $P, Q, R, S$  and let  $O'$  be the centre of the circle through  $P, Q, R, S$ .

Now  $OP = OR, O'P = O'R$ .

The triangles  $OPB, OQB$  are equal and the triangles  $OQC, ORC$  are equal.

Therefore the angle  $OPB =$  the angle  $OQB$   
 $=$  the supplement of  $OQC$   
 $=$  the supplement of  $ORC$   
 $=$  the angle  $ORD$ .

Similarly the angle  $O'PB =$  the angle  $O'RD$ .

Therefore while the triangle  $OPO'$  rotates round  $OO'$  into the position  $ORO'$ , the line  $APB$  rotates into a position coincident in direction with  $CRD$ .

Similarly by rotation round  $OO'$  either of the sides  $BQC, DSA$  might be rotated into the position of the other. Any straight line obtained by the rotation of any one of the sides of the quadrilateral about  $OO'$  touches every one of the series of spheres which touch the sides of the quadrilateral and have their centres in  $OO'$ .

And every straight line obtained by rotation of either of the opposite sides  $AB, CD$  about  $OO'$  intersect every straight line that can be obtained by rotation of either of the opposite sides  $BC, DA$  about  $OO'$ .

We have now proved that, if in a skew quadrilateral any one of the three equations

$$\begin{aligned} AB + BC - CD - DA &= 0, \\ AB - BC + CD - DA &= 0, \\ AB - BC - CD + DA &= 0, \end{aligned}$$

be satisfied by the sides, there exists a corresponding series of spheres touching the sides.

If two of these relations be satisfied, by two pairs of sides being equal, there exist two series of spheres.

If the three equations be simultaneously satisfied, by all the sides being equal, there exist three series of spheres.

## INTRASCIBED SPHERE OF TETRAHEDRON.

Consider a tetrahedron  $ABCD$  in which the sums of the pairs of opposite edges are equal,

that is

$$AB + CD = AC + BD = AD + CB.$$

We have already proved that it is possible to describe a series of spheres to touch the four sides of the quadrilateral  $ABCD$ , and that the centres of all the spheres lie on a fixed straight line  $OO'$ . Draw a plane through  $B$  at right angles to the external bisector of the angle  $ABD$ . This plane will meet the line  $OO'$  in some point. Let it meet it in  $O$ .  $O$  is the centre of a sphere which touches the sides  $AB$ ,  $BC$ ,  $CD$ ,  $DA$ , and  $BD$  in  $P$ ,  $Q$ ,  $R$ ,  $S$ ,  $T$ .

Because  $CA + BD = AB + CD$  and  $BT = BP$  and  $DT = DR$ ;

therefore  $CA = CR + AP$ .

Therefore  $CA$  is a tangent to the sphere (§ 7, p. 592).

Hence a sphere can be described to touch the six edges of a tetrahedron, in which the sums of pairs of opposite edges are equal.

This sphere will pass through the inscribed circle of each face of the tetrahedron.

This may be called an intrascribed sphere.

Similarly it may be proved that a sphere can be described to touch the six edges of a tetrahedron, in which the differences of opposite edges are equal.

This sphere will pass through the inscribed circle of that face of the tetrahedron, whose edges are either all not greater than or all not less than the opposite edges of the tetrahedron; and it will also pass through an escribed circle of each of the other faces.

In the case of a tetrahedron, whose opposite edges are equal, four such spheres can be described.

These may be called extrascibed spheres.

A regular tetrahedron has four such extrascibed spheres and also an intrascribed sphere.

## EULER'S EQUATION.

There is a remarkable relation of a very simple nature connecting the numbers of the faces, the edges and the corners of every closed polyhedron. It is due to Euler\*.

It may be stated as follows:

If  $F$ ,  $E$ ,  $C$  be the numbers of the faces, of the edges and of the corners of any closed polyhedron,

then  $F + C = E + 2$ .

It may be proved as follows:

Take any closed polyhedron, say of  $F$  faces.

Remove one face without interfering with any other faces:

this will not destroy any edge or any corner of the polyhedron:

next remove another face adjoining the opening caused by the removal of the previous face:

continue this operation repeatedly until there is no face left:

at the removal of every face, except the first and the last, the number of edges destroyed is greater by one than the number of corners destroyed:

therefore the total number of edges destroyed by the removal of all the  $F$  faces is greater by  $F - 2$  than the total number of corners destroyed,

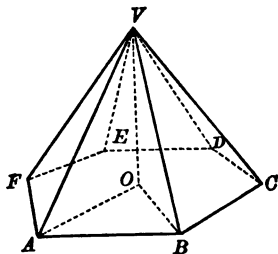
$$\text{that is } E = C + F - 2$$

$$\text{or } F + C = E + 2.$$

\* Leonard Euler, born at Basle 1707, died at St Petersburg 1783.

## REGULAR SOLID ANGLES.

If we take any regular polygon of  $n$  sides and through  $O$  the centre of the circumscribed circle draw a straight line  $OV$  at right



angles to the plane of the polygon, then  $V(ABC\dots)$  will be a regular  $n$  hedral angle.

The angle  $AVB$  is less than the angle  $AOB$ . (Ex. 1, p. 551.)

Now the angle subtended by a side of a regular polygon of  $n$  sides at the centre of the circumscribed circle is  $\frac{1}{n}$ th part of four right angles.

If the polygon be a triangle  $ABC$ , the angle  $AOB$  is  $\frac{1}{3}$ rd of four right angles or four-thirds of a right angle; therefore the angle  $AVB$  might be the angle of an equilateral triangle, of a square or of a regular pentagon which are respectively

two-thirds of a right angle, a right angle, and six-fifths of a right angle; but it could not be the angle of a regular polygon of six or more sides.

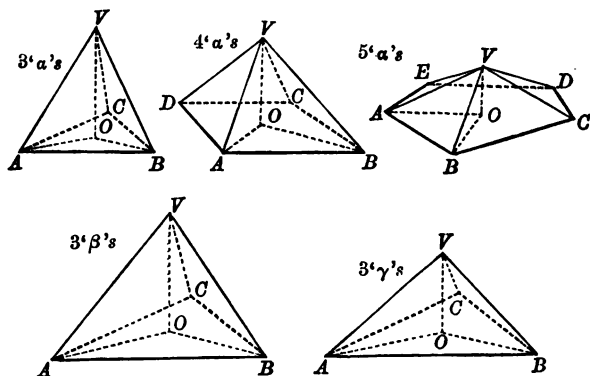
If the polygon be a square  $ABCD$ , the angle  $AOB$  is a right angle, and the angle  $AVB$  might be the angle of an equilateral triangle, but could not be the angle of any regular polygon of more than three sides.

If the polygon be a regular pentagon  $ABCDE$ , the angle  $AOB$  is equal to four-fifths of a right angle, and the angle  $AVB$  might be the angle of an equilateral triangle, but not of any regular polygon of more than three sides.



If the polygon be a regular hexagon  $ABCDEF$  or a regular polygon of more than six sides the angle  $AOB$  is equal to, or less than, two-thirds of a right angle, and the angle  $AVB$  cannot be equal to the angle of any regular polygon whatever.

We have proved therefore that there are but five different forms of regular solid angles of which the plane angles are equal to those of the angles of a regular polygon.



This proves that it is not possible to construct more than five different forms of regular polyhedrons.

It has been just shewn that the only possible regular polyhedrons are those which have their solid angles composed of plane angles as follows,

$3 \text{ "}\alpha\text{"}$ s,  $4 \text{ "}\alpha\text{"}$ s,  $5 \text{ "}\alpha\text{"}$ s,  $3 \text{ "}\beta\text{"}$ s,  $3 \text{ "}\gamma\text{"}$ s,

where  $\alpha$  is the angle of an equilateral triangle,

$\beta$  a right angle (the angle of a square),

and  $\gamma$  the angle of a regular pentagon.

We will now assume that such figures are possible and we will shew how to find the number of faces, corners and edges of each.

## THE REGULAR SOLIDS.

Let  $F$  be the number of faces,  $C$  the number of corners and  $E$  the number of edges.

First take 3 "a"s.

Since  $F$  is the number of faces and each face is a triangle, the number of plane angles is  $3F$ .

Since three plane angles occur at each corner the number of corners is one-third of the number of plane angles, i.e.  $\frac{1}{3}(3F)$ ;

therefore  $C = F$ .

Since the number of sides possessed by  $F$  separate triangles is  $3F$  and each edge forms a side of two triangles; therefore the number of edges is one-half of  $3F$ ;

therefore  $E = \frac{3}{2}F$ .

Again, by Euler's equation  $F + C = E + 2$ ;

therefore  $F + F = \frac{3}{2}F + 2$ ,

that is  $F = 4$ ,  $C = 4$  and  $E = 6$ .

In a similar way may be found the numbers of faces, corners and edges of each of the other four regular solids. The results are given in the following Table.

| Name of Figure | Plane Angles at a Corner. | Sides of a Face. | Number of Faces | Number of Corners | Number of Edges |
|----------------|---------------------------|------------------|-----------------|-------------------|-----------------|
| Tetrahedron    | 3                         | 3                | 4               | 4                 | 6               |
| Cube           | 3                         | 4                | 6               | 8                 | 12              |
| Octahedron     | 4                         | 3                | 8               | 6                 | 12              |
| Dodecahedron   | 3                         | 5                | 12              | 20                | 30              |
| Icosahedron    | 5                         | 3                | 20              | 12                | 30              |

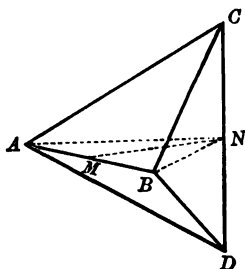
It will be observed from this Table that

- (1) the reciprocal figure of a Tetrahedron is a Tetrahedron,
- (2) the reciprocal figure of a Cube is an Octahedron and vice versa,
- (3) the reciprocal figure of a Dodecahedron is an Icosahedron and vice versa. (See p. 607.)

We will now proceed to shew how to construct each of the five regular solids with its edge equal to a given straight line and thus prove the possibility of their existence.

## THE REGULAR TETRAHEDRON.

Draw a straight line  $AB$  of length  $a$ , bisect  $AB$  in  $M$ , draw  $MN$  at right angles to  $AB$  such that  $MN^2 = \frac{1}{3}a^2$ . Through  $N$  draw  $CND$



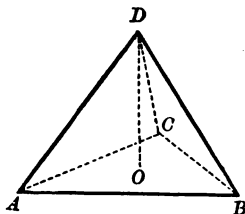
at right angles to the plane  $NAB$  and take  $NC$ ,  $ND$  each equal to  $\frac{1}{2}a$ . Then  $A$ ,  $B$ ,  $C$ ,  $D$  are the corners of a regular tetrahedron.

PROOF.  $AC^2 = AM^2 + MN^2 + NC^2$  (§ 4, p. 586).  
 $= \frac{1}{4}a^2 + \frac{1}{3}a^2 + \frac{1}{4}a^2 = a^2$ ;  
 therefore  $AC = a = AB$ .

Similarly the other edges may be proved equal to either  $AB$  or  $CD$ ; therefore each face is an equilateral triangle, and all the solid angles are equal. (Add. Prop. 1.)

The following is an alternative construction :—

Construct the equilateral triangle  $ABC$ . Find  $O$  the centre of the circle  $ABC$ . Draw  $OD$  at right angles to the plane  $ABC$ .

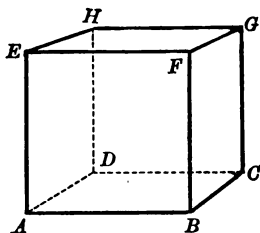


Find  $D$  in  $OD$  such that  $AD$  is equal to  $AB$ .

Then as before  $A$ ,  $B$ ,  $C$ ,  $D$  are the corners of the tetrahedron.

## THE REGULAR HEXAHEDRON OR CUBE.

In any plane construct a square  $ABCD$  such that  $AB$  is equal to  $a$ , and in a plane at right angles construct the square  $ABFE$ ;



through  $CD$  draw the plane  $CDHG$  parallel to  $ABFE$ , and through  $EF$  draw the plane  $EFGH$  parallel to  $ABCD$ .

Draw the planes  $BCF$  and  $ADE$  to meet  $GH$  in  $G$  and  $H$  respectively.

Then  $ABCDEFGH$  is a cube.

## PROOF.

The two planes  $ADHE$ ,  $BCGF$  are parallel (Prop. 15);

hence the figure is a parallelepiped. (Def. 13.)

Each of the plane angles at  $A$  is a right angle;

hence the figure is an orthohedron. (Def. 13 and Prop. 18.)

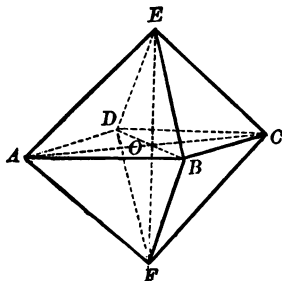
The three edges meeting at  $A$  are equal;

therefore the figure  $ABCDEFGH$  is a cube. (Def. 13.)

## THE REGULAR OCTAHEDRON.

Construct a square  $ABCD$  whose side is equal to  $a$ ; draw  $AC$ ,  $BD$  meeting in  $O$ ; through  $O$  draw a straight line  $EOF$  at right angles to the plane  $ABCD$  and make  $OE$ ,  $OF$  each equal to  $OA$ .

Draw the plane faces  $EAB$ ,  $EBC$ ,  $ECD$ ,  $EDA$ ,  $FAB$ ,  $FBC$ ,  $FCD$ ,  $FDA$ .



Then  $EABCDF$  is a regular octahedron.

PROOF. In the right-angled triangles  $AOE$ ,  $AOB$ ,  
 $AO$ ,  $OE$  are equal to  $AO$ ,  $OB$ ;  
 therefore  $AE$  is equal to  $AB$ .

Similarly it can be shewn that each of the other edges is equal to one of the sides of the square  $ABCD$ .

Hence each of the faces of the figure is an equilateral triangle.

The solid angle at each corner is composed of four equal solid angles, for instance the tetrahedral angle  $B$  ( $AECE$ ) is made up of four trihedral angles  $B(AEO)$ ,  $B(ECO)$ ,  $B(CFO)$ , and  $B(FAO)$ , each of which is contained by three plane angles one of which is two-thirds and the other two each one-half of a right angle. The solid angles of the figure  $ABCDEF$  are therefore equal.

Hence the figure is a regular octahedron.

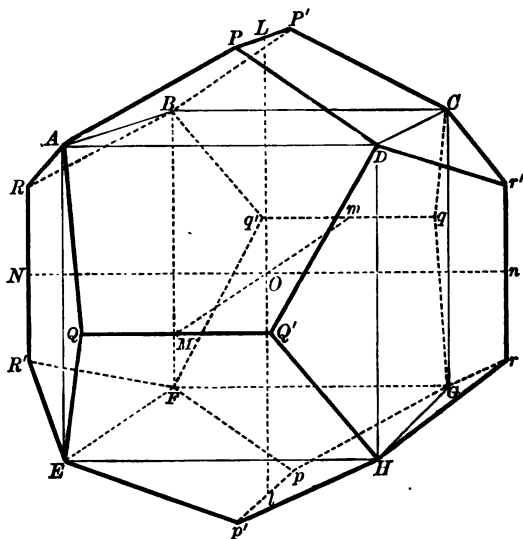
## THE REGULAR DODECAHEDRON.

Let  $l$  be equal to the diagonal of a regular pentagon, whose side is equal to  $a$ , in which case,

by Ptolemy's Theorem, we have  $l^2 = al + a^2$ . (III. Prop. 37 B.)

Construct a cube  $ABCDEFGH$ , whose edge is equal to  $l$ .

Let  $O$  be its centre, draw from  $O$ ,  $OL$ ,  $OM$ ,  $ON$  at right angles to the faces  $ABCD$ ,  $ADHE$ ,  $AEFB$  respectively.



And take  $OL$ ,  $OM$ ,  $ON$  each equal to  $\frac{1}{2}(a+l)$ ; draw  $PLP'$ ,  $QM'Q'$ ,  $RNR'$ , parallel to  $AB$ ,  $AD$ ,  $AE$  respectively.

Take  $LP$ ,  $LP'$ ,  $MQ$ ,  $MQ'$ ,  $NR$ ,  $NR'$  each equal to  $\frac{1}{2}a$ .

Let  $l$ ,  $m$ ,  $n$ ,  $p$ ,  $p'$ ,  $q$ ,  $q'$ ,  $r$ ,  $r'$  be the reflexes in  $O$  of  $L$ ,  $M$ ,  $N$ ,  $P$ ,  $P'$ ,  $Q$ ,  $Q'$ ,  $R$ ,  $R'$ . (See p. 620.)

Then the twenty points  $A$ ,  $B$ ,  $C$ ,  $D$ ,  $E$ ,  $F$ ,  $G$ ,  $H$ ,  $P$ ,  $P'$ ,  $Q$ ,  $Q'$ ,  $R$ ,  $R'$ ,  $p$ ,  $p'$ ,  $q$ ,  $q'$ ,  $r$ ,  $r'$  are the corners of a regular dodecahedron.

PROOF.

The projections of  $AP$  on  $AB$ ,  $AD$ ,  $AE$  are equal to  $\frac{1}{2}(l-a)$ ,  $\frac{1}{2}l$ ,  $\frac{1}{2}a$ .  
(p. 533.)

Therefore  $AP^2 = \frac{1}{4}(l-a)^2 + \frac{1}{4}l^2 + \frac{1}{4}a^2 = \frac{1}{4}(l^2 - al + a^2) = a^2$ .  
(§ 4, p. 586.)

Therefore  $AP$  is equal to  $a$ .

Similarly every edge of the figure is equal to  $a$ .

The quadrilateral  $APP'B$  has sides  $AP$ ,  $PP'$ ,  $P'B$ ,  $BA$  equal to  $a$ ,  $a$ ,  $a$ ,  $l$  respectively and since  $PP'$  is parallel to  $AB$  it forms part of a regular pentagon whose side is equal to  $a$ .

The triangle  $ARB$  has sides  $AR$ ,  $RB$ ,  $BA$  equal to  $a$ ,  $a$ ,  $l$  respectively and therefore is equal to the remaining part of the pentagon.

Let  $S$  be the middle point of  $AB$ .

The projections of  $RS$  and  $RL$  on  $AD$  are equal to  $\frac{1}{2}a$  and  $\frac{1}{2}(a+l)$ , and their projections on  $AE$  are equal to  $\frac{1}{2}(l-a)$  and  $\frac{1}{2}l$ ,  
and

$$a : a+l :: l-a : l,$$

$$\text{since } al = l^2 - a^2.$$

Therefore  $RSL$  is a straight line and the figures  $APP'B$ ,  $ARB$  lie in one plane and therefore the figure  $PARBP'$  is a regular pentagon.

Similarly each face of the figure is a regular pentagon; and, as each solid angle of the figure is contained by three plane angles of the same size, they are all equal; therefore the figure is a regular dodecahedron.

## THE REGULAR ICOSAHDREDON.

Let  $l$  be equal to the diagonal of a regular pentagon, whose side is equal to  $a$ , in which case,

by Ptolemy's Theorem, we have  $l^2 = al + a^2$ . (III. Prop. 37 B.)

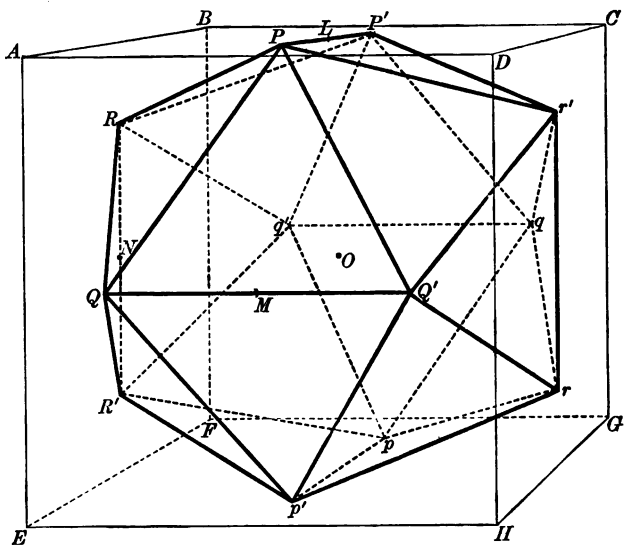
Construct a cube  $ABCDEFGH$  whose edge is equal to  $l$ .

Let  $O$  be the centre of the cube. Draw perpendiculars  $OL$ ,  $OM$ ,  $ON$  on the three faces  $ABCD$ ,  $ADHE$ ,  $AEFB$ , and in these faces draw lines  $PLP'$ ,  $QMQ'$ ,  $RNR'$ , parallel to  $AB$ ,  $AD$ ,  $AE$  respectively.

Take  $LP$ ,  $LP'$ ,  $MQ$ ,  $MQ'$ ,  $NR$ ,  $NR'$  each equal to  $\frac{1}{2}a$ .

Let  $p$ ,  $p'$ ,  $q$ ,  $q'$ ,  $r$ ,  $r'$  be the reflexes of  $P$ ,  $P'$ ,  $Q$ ,  $Q'$ ,  $R$ ,  $R'$  in the point  $O$ . (p. 620.)

The twelve points  $P$ ,  $P'$ ,  $Q$ ,  $Q'$ ,  $R$ ,  $R'$ ,  $p$ ,  $p'$ ,  $q$ ,  $q'$ ,  $r$ ,  $r'$  are the corners of a regular icosahedron.



The projections of  $PQ$  upon  $AB$ ,  $AD$ ,  $AE$  are equal to  $\frac{1}{2}(l-a)$ ,  $\frac{1}{2}a$ ,  $\frac{1}{2}l$  respectively. (p. 533.)



Therefore  $PQ^2 = \frac{1}{4}(l-a)^2 + \frac{1}{4}a^2 + \frac{1}{4}a^2 = \frac{1}{2}(l^2 - al + a^2) = a^2$ .  
(§ 4, p. 586.)

Therefore  $PQ$  is equal to  $a$ . Similarly every other edge of the figure may be proved equal to  $a$ .

Therefore every face of the figure is an equilateral triangle.

And all the points lie on a sphere whose centre is  $O$ ,

$$\text{since } OP^2 = \frac{1}{4}(a^2 + l^2).$$

Each solid pentahedral angle is composed of five equal trihedral angles; for instance the pentahedral angle  $P(P'r'Q'QR)$  is composed of the five equal trihedral angles  $P(P'r'O)$ ,  $P(r'Q'O)$ ,  $P(Q'OQ)$ ,  $P(QRO)$ ,  $P(RP'O)$ .

And therefore all the solid angles of the figure are equal and the figure is a regular icosahedron.

Each of these regular solids has a circumscribed sphere passing through all the angular points, an inscribed sphere touching all the faces, and an intrascribed sphere touching all the edges. The squares on the radii of the circumscribed, the inscribed and the intrascribed spheres, which we will call  $R$ ,  $r$ ,  $\rho$  respectively are given in the following Table.

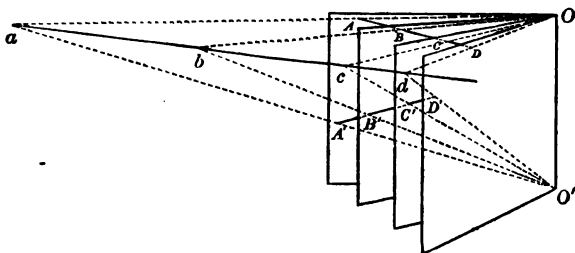
|              | $4R^2$           | $4r^2$                     | $4\rho^2$        |
|--------------|------------------|----------------------------|------------------|
| Tetrahedron  | $\frac{3}{2}a^2$ | $\frac{1}{6}a^2$           | $\frac{1}{2}a^2$ |
| Cube         | $3a^2$           | $a^2$                      | $2a^2$           |
| Octahedron   | $2a^2$           | $\frac{2}{3}a^2$           | $a^2$            |
| Dodecahedron | $3l^2$           | $\frac{1}{3}(2a+3l)(2a+l)$ | $3l^2 - a^2$     |
| Icosahedron  | $a^2 + l^2$      | $l^2 - \frac{1}{3}a^2$     | $l^2$            |

In this Table  $a$  is equal to the edge of the figure, and  $l$  satisfies the equation  $l^2 = al + a^2$ .

### ANHARMONIC PROPERTY OF FOUR PLANES.

*The four points in which four fixed planes which pass through a straight line are cut by any other straight line form an anharmonic range of constant ratio.*

Let  $OAA'$ ,  $OBB'$ ,  $OCC'$ ,  $ODD'$  be four planes passing through a common straight line  $OO'$  and cutting two straight lines  $ABCD$ ,  $A'B'C'D'$ .



Take any two points  $O, O'$  in  $OO'$  and draw the planes  $OABCD$ ,  $O'A'B'C'D'$ ,

and let  $OA, O'A'$  meet in  $a$ ;  $OB, O'B'$  in  $b$ ;

$OC, O'C'$  in  $c$  and  $OD, O'D'$  in  $d$ .

$a, b, c, d$  lie in a straight line, the common section of the planes  $OABCD, O'A'B'C'D'$ .

Now  $O(ABCD)$  is an anharmonic pencil cut by two transversals  $ABCD, abcd$ .

Therefore  $ABCD, abcd$  are two like anharmonic ranges.

(VI. Add. Prop. 13.)

And  $O'(A'B'C'D')$  is an anharmonic pencil cut by two transversals  $A'B'C'D', abcd$ .

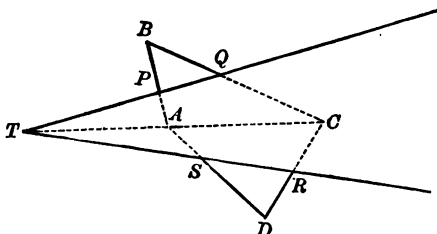
Therefore  $A'B'C'D', abcd$  are two like anharmonic ranges, therefore  $ABCD, A'B'C'D'$  are two like anharmonic ranges.

EXTENSION OF THE THEOREM OF MENELAUS.

If a plane cut the four sides of a skew quadrilateral, produced if necessary, the ratio compounded of the ratios of the segments of the sides taken in order is equal to unity.

Let a plane  $PQRS$  cut the sides  $AB$ ,  $BC$ ,  $CD$ ,  $DA$  of a skew quadrilateral  $ABCD$  in  $P$ ,  $Q$ ,  $R$ ,  $S$  respectively.

Let the plane  $PQRS$  cut the two planes  $ABC$ ,  $ADC$  in the straight lines  $TPQ$ ,  $TSR$ ; they necessarily intersect in some point  $T$  in  $AC$ . (Prop. 3.)



Because  $TPQ$  is a transversal of the triangle  $ABC$ , the ratio compounded of the ratios  $AP$  to  $PB$  and  $BQ$  to  $QC$  is equal to the ratio  $AT$  to  $TC$ : (VI. Add. Prop. 1.)

and because  $TSR$  is a transversal of the triangle  $ADC$ , the ratio compounded of the ratios  $CR$  to  $RD$  and  $DS$  to  $SA$  is equal to the ratio  $CT$  to  $TA$ .

Therefore the ratio compounded of the ratios  $AP$  to  $PB$ ,  $BQ$  to  $QC$ ,  $CR$  to  $RD$ , and  $DS$  to  $SA$  is equal to the ratio compounded of the ratios  $AT$  to  $TC$  and  $CT$  to  $TA$ , that is, to unity.

It follows from the result just proved that if  $AP$  be to  $PB$  as  $DR$  to  $RC$ ,

then  $AS$  is to  $SD$  as  $BQ$  to  $QC$ .

Now if the three lines  $AD$ ,  $PR$ ,  $BC$  be parallel to the same plane:

then  $AP$  is to  $PB$  as  $DR$  to  $RC$ , (Prop. 17.)

and if  $AS$  be to  $SD$  as  $BQ$  to  $QC$ , the three lines  $AB$ ,  $QS$ ,  $DC$  are parallel to one plane. (Ex. 2 on Prop. 17.)

We may therefore generalize this proposition thus,

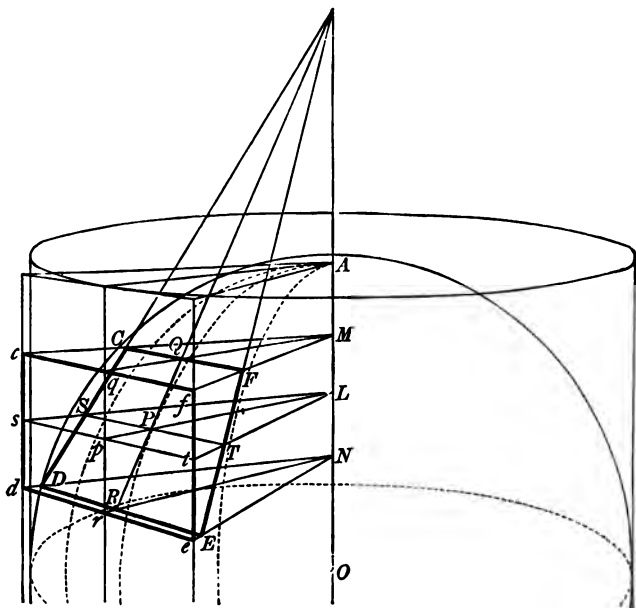
Every straight line which can be drawn to intersect three given non-intersecting straight lines which are all parallel to a fixed plane is parallel to a second fixed plane.

## SURFACE OF SPHERE.

Let  $AB$  be a diameter of a sphere whose centre is  $O$ , and let  $APB$  be a great circle. Let  $P$  be the middle point of  $QPR$  a tangent to the circle  $APB$ . Through  $P$ ,  $Q$  and  $R$  let planes  $LST$ ,  $MCF$ ,  $NDE$  be drawn at right angles to  $AB$ , cutting  $AB$  in  $L$ ,  $M$  and  $N$ . Let the tangent plane to the sphere at  $P$  cut these planes in  $ST$ ,  $CF$  and  $DE$ , and let two planes through  $AB$  equally inclined to  $APB$  cut the tangent plane at  $P$  in  $CD$  and  $FE$ .

Now the area of the quadrilateral  $CDEF$  is equal to  $QR \cdot ST$ .

Let  $LP$  produced cut the enveloping cylinder of the sphere, whose generating lines are parallel to  $AB$ , in  $p$ ; and the lines  $MC$ ,  $ND$ ,  $NE$ ,  $MF$ ,  $MQ$ ,  $NR$ ,  $LS$ ,  $LT$  cut the tangent plane to the cylinder at  $p$  in  $c$ ,  $d$ ,  $e$ ,  $f$ ,  $q$ ,  $r$ ,  $s$ ,  $t$ .



Now the area of the quadrilateral  $cdef$  is equal to  $qr \cdot st$ .

And  $QR : qr = PO : PL = pL : PL = st : ST$ .

Therefore  $QR \cdot ST$  is equal to  $qr \cdot st$ ;

or, the quadrilaterals  $CDEF$ ,  $cdef$  are equal in area.

Now suppose the surface of the sphere to be divided into any number of lunes of equal angles by planes drawn through  $AOB$ , and between the planes forming each of these lunes let a quadrilateral be described corresponding to and equal to the quadrilateral  $CDEF$ ; and also corresponding to and equal to the rectangle  $cdef$ .

If the number of the lunes be increased, the quadrilaterals  $CDEF$ ... can be made to approach as nearly as we like to a strip of the surface of a cone whose axis is in  $AOB$ , which touches the sphere at  $P$ , and which is generated by the rotation of the line  $QR$  round  $AOB$  through four right angles, and at the same time the quadrilaterals  $cdef$ ... approach as nearly as we like to a strip of the surface of the enveloping cylinder.

Hence the curved surface of the portion of a right circular cone, between two parallel planes at right angles to the axis, is equal to the surface of the portion of a co-axial circular cylinder, intercepted by the same planes; where the radius of the cylinder is equal to the radius of a sphere which touches the cone at points equidistant from the parallel planes.

Let a polygon of an even number of sides be described about the circle  $APB$  touching at  $A$  and  $B$  and let  $P$  be the point of contact of one of the sides  $QPR$ .

By rotation round  $AOB$  a strip of a cone will be generated by each of the other sides of the polygon, except those which touch at  $A$  and  $B$ .

The surface of all these strips of cones will be equal to the surface of the cylinder.

Now if the number of the sides of the polygon be increased the strips of the cones can be made to approach as near as we like to the surface of the sphere, hence the surface of the sphere is equal to the curved surface of the cylinder.

Similarly it may be proved that if two closed curves be drawn on the sphere and on the cylinder so that the straight line joining corresponding points on the two curves always cuts the axis of the cylinder at right angles, the surfaces of the portions of the sphere and the cylinder bounded by these curves are equal.

## VOLUME OF SPHERE.

It has been proved that the quadrilaterals  $CDEF$ ,  $cdef$  are equal in area.

The volume of the pyramid  $OCDEF$  is equal to  $\frac{1}{3}OP \cdot CDEF$ , because  $OP$  is perpendicular to  $CDEF$ .

And the volume of the prism  $McfNde$  is equal to half the orthohedron, whose base is  $cdef$  and height  $Lp$ , that is, equal to  $\frac{1}{3}OP \cdot CDEF$ .

Hence the volume of  $OCDEF$  is two-thirds of the volume of  $McfNde$ .

Again, if lunes be drawn as before and between the planes forming each lune a pyramid and a prism be constructed corresponding to  $OCDEF$ ,  $McfNde$  respectively and then the number of the lunes be increased indefinitely, the volume generated by the rotation of the triangle  $OQR$  round  $AOB$  is two-thirds of the volume of the cylinder generated by the rotation of  $qrNM$  round  $AOB$ .

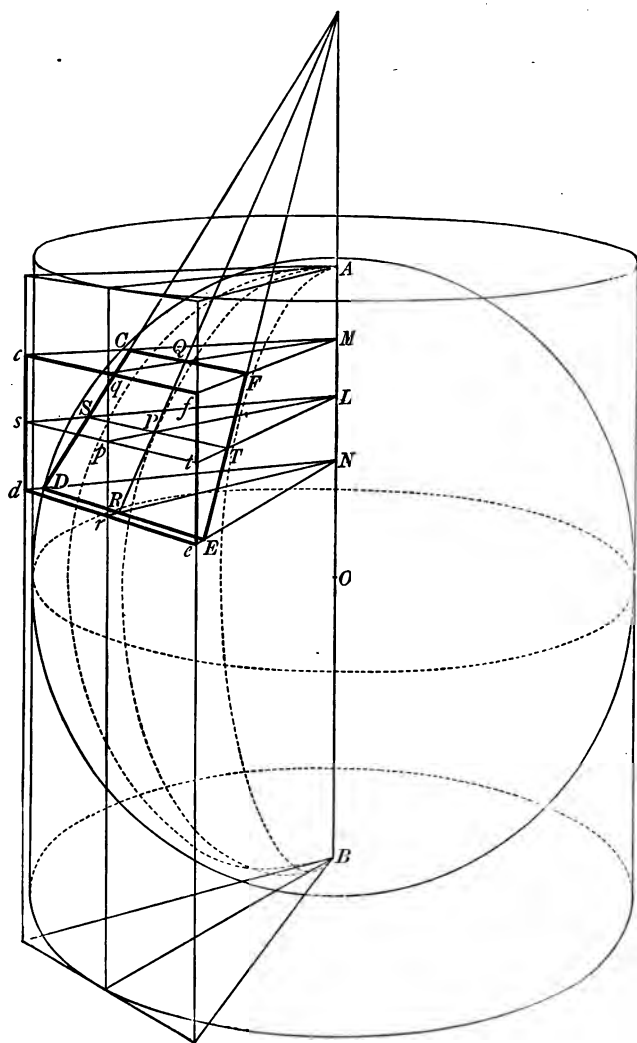
Again, if a polygon of an even number of sides be described in the plane  $APB$  as before and a strip of a cone generated by the rotation of each of its sides, except those which touch at  $A$  and  $B$ , the volume generated by the rotation of the polygon omitting the part formed by the two triangles, whose vertices are at  $O$  and whose bases are the tangents which touch at  $A$  and  $B$ , is two-thirds of the volume of the cylinder.

Again, if the number of sides of the polygon be increased indefinitely, the volume generated by that part of the polygon becomes equal to the volume of the sphere.

Hence the volume of the sphere is two-thirds of the volume of the enveloping cylinder.

Similarly it may be proved that, if two closed curves be drawn on the sphere and on the cylinder so that the straight line joining corresponding points on the two curves always cuts the axis of the cylinder at right angles, the volume of the portion of the sphere enclosed by radii of the sphere passing through the curve on the sphere is two-thirds of the volume of the portion of the cylinder enclosed by straight lines drawn at right angles to the axis passing through the curve on the cylinder.

The relations which we have established between the surfaces and the volumes of the sphere and the enveloping cylinder were discovered by Archimedes, a Greek philosopher, who was killed at the siege of Syracuse, 212 B.C.



## MISCELLANEOUS EXERCISES.

1. If a straight line be parallel to each of two planes it is parallel to their common section.

2. The perpendicular drawn from a vertex of a regular tetrahedron on the opposite face is three times that drawn from its own foot on any of the other faces.

3. A pyramid stands on an equilateral triangle as base and the angles at the vertex are right angles: shew that the sum of the perpendiculars on the faces from any point of the base is constant.

4. The edges  $AB$ ,  $CD$  of a tetrahedron are at right angles, and  $K$ ,  $K'$  are the orthocentres of the faces  $ABC$ ,  $ABD$ . Prove that  $KK'$ ,  $AB$  are at right angles.

5. If  $OABC$  be a tetrahedron, prove geometrically that it is impossible for the opposite edges  $OA$ ,  $BC$ , and  $OB$ ,  $CA$  and  $OC$ ,  $AB$  to be inclined to one another at the same angle unless that angle be a right angle. The order in which the letters are given is to be considered as fixing which angle between each pair of lines is considered.

6. If  $ABCD$  be an orthocentric tetrahedron and  $O$  its orthocentre, then the five tetrahedrons formed by any four of the five points  $O$ ,  $A$ ,  $B$ ,  $C$ ,  $D$  is an orthocentric tetrahedron and the fifth point is the orthocentre.

7. If  $ABCD$  be an orthocentric tetrahedron and  $O$  its orthocentre, the twelve-point spheres of the five tetrahedrons formed by joining the points  $O$ ,  $A$ ,  $B$ ,  $C$ ,  $D$  intersect each other in pairs in the nine-point circles of the ten triangles formed by joining the points  $O$ ,  $A$ ,  $B$ ,  $C$ ,  $D$ .

8. In an orthocentric tetrahedron each solid angle is contained by three plane angles of the same kind, i.e. all acute, all right, or all obtuse.

9. In a tetrahedron  $ABCD$ ,  $AB$  is at right angles to  $CD$ . Also the sum of the inclinations of the planes meeting in  $AB$ , and of the planes meeting in  $CD$  is equal to two right angles; and the planes bisecting these angles meet in the shortest distance  $EF$  between  $AB$  and  $CD$ . Prove that the rectangle  $AB$ ,  $CD$  is four times the square on  $EF$ .

10.  $ABCD$  is a given tetrahedron,  $A'$ ,  $B'$ ,  $C'$  points on  $DA$ ,  $DB$ ,  $DC$  taken so that the planes  $ABC$ ,  $A'B'C'$  are parallel; prove that the locus of the intersection of the planes  $AB'C'$ ,  $BC'A'$ ,  $CA'B'$  is the line joining  $D$  to the centre of gravity of the triangle  $ABC$ .

11. A point moves so that the sum of the volumes of the tetrahedrons formed by joining it to the angular points of two given triangles is constant: prove that its locus is a plane.



12. The section of a rectangular parallelepiped made by a plane passing through one of its edges is a rectangle.

13. The middle points of the six edges of a cube which do not meet a diagonal of the cube lie in a plane which bisects the diagonal and is at right angles to it.

14. Every triangular section of a rectangular parallelepiped is an acute-angled triangle.

15. In a parallelepiped,  $AB$ , the diagonal of one face is at right angles to the edges of intersection of the four faces not parallel to the first. A similar fact is true for  $AC$ ,  $AD$ , the diagonals of the other faces meeting at  $A$ .

Shew that the parallelepiped is rectangular.

16. The section of a parallelepiped by every plane through the centre of the parallelepiped is either a parallelogram or a hexagon whose opposite sides are equal and parallel.

17. The sum of squares on the diagonals of a parallelepiped is equal to the sum of the squares on the edges.

18. Prove that, if  $OP$  be a diagonal of a parallelepiped, and  $OA$ ,  $OB$ ,  $OC$  three edges, then the sum of the squares on  $OP$ ,  $OA$ ,  $OB$ ,  $OC$  is equal to the sum of the squares on  $PA$ ,  $PB$ ,  $PC$ .

19. An octahedron can be formed of any eight equal triangles provided that they are acute angled.

20. Two intersecting circles cut at the same angle as their inverse circles.

21. Prove that a sphere can be described through any two circles which intersect in two points. What are the conditions that a sphere can be described through two non-intersecting circles?

22. If six points  $A$ ,  $B$ ,  $C$ ,  $D$ ,  $E$ ,  $F$  be such that three circles  $ABCD$ ,  $CDEF$ ,  $EFAB$  may be described through them in fours, the six points lie on a sphere. What are the conditions that a sphere can be described through three non-intersecting circles?

23. A straight line is divided at a fixed point into parts of a constant length, and the ratios of the distances of its extremities from two other fixed points are equal: prove that the line always lies in the same plane.

24. Shew how to reduce the Solid problem to describe a sphere to touch two given planes and to pass through two given points to the Plane problem to find a point on a given straight line equidistant from a given point and another given straight line.

25. In a tetrahedron of which the opposite edges are equal, shew that (i) the centre of gravity, the centre of the inscribed sphere and the centre of circumscribing sphere all coincide, (ii) the shortest distance between opposite edges bisects those edges, and (iii) that the three shortest distances are all mutually at right angles and are bisected in the centre of gravity.

26. Each edge of a tetrahedron is equal to the opposite edge : prove that the sum of the squares on the sides of a face is double the square on the diameter of the circumscribed sphere.

27. It is possible to describe two circular cones such that each passes through two given circular sections of the same sphere. The vertices of the two cones lie on the polar line of the common section of the planes of the two circles.

28. Every tetrahedron, self-conjugate\* with respect to a sphere, is orthocentric.

29. An orthocentric tetrahedron  $ABCD$  and its orthocentre  $O$  are such that each of the five tetrahedrons is self-conjugate with respect to a sphere, whose centre is at the fifth point. These five spheres cut each other at right angles. One sphere has an imaginary, and the other four real radii.

30. The six spheres, described on the edges of a self-conjugate tetrahedron as diameters, all cut the sphere at right angles.

31. If two points  $P, Q$  be conjugate with respect to a circle in a plane in which they lie, they are conjugate with respect to any sphere passing through the circle.

32. If  $PQR$  be a self-conjugate triangle with respect to the circular section of a sphere made by the plane  $PQR$ , and if  $S$  be the pole of the plane  $PQR$  with respect to the sphere, the tetrahedron  $PQRS$  is self-conjugate with respect to the sphere.

33. In a self-conjugate tetrahedron one vertex is within the sphere and the other three without. All the plane angles of the tetrahedron at the internal vertex are obtuse angles.

34. If  $PQRS$  be a self-conjugate tetrahedron, a cone, whose vertex is  $R$  or  $S$  drawn through a circle, whose plane passes through  $PQ$ , will cut the sphere again in a second circle, whose plane also passes through  $PQ$ .

35. Prove that any two, reverse tetrahedrons may be divided each into six parts, which in pairs are equal in all respects, each part being composed of two tetrahedrons and the twelve tetrahedrons having a common vertex at the centre of the inscribed sphere.

36. If  $\alpha, \beta, \gamma, \delta$  be the centroids of the faces of the tetrahedron  $ABCD$  opposite  $A, B, C, D$ , and if  $O$  be the centre of the circle  $ABC$ ,  $H$  the foot of the perpendicular from  $D$  on the plane  $ABC$ , and  $P$  the orthocentre of the triangle  $ABC$ , then the centre of the section of the sphere  $\alpha\beta\gamma\delta$  by the plane  $ABC$  is the centroid of the triangle  $OHP$ .

\* Two points are said to be conjugate with respect to a circle or a sphere, when each point is on the polar line or the polar plane of the other point; a triangle or a tetrahedron is said to be self-conjugate with respect to a circle or a sphere when each vertex is on the polar line or the polar plane of each of the other vertices.

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